Questions Let $f(x)$, $g(x)$, $h(x)$ be functions with

$$\lim_{x\to 2} f(x) = \infty, \quad \lim_{x\to 2} g(x) = \infty, \quad \lim_{x\to 2} h(x) = 5.$$

1. $\lim_{x\to 2} f(x) + g(x) = \underline{\quad \infty \quad}$

2. $\lim_{x\to 2} -f(x) = \underline{\quad -\infty \quad}$

3. $\lim_{x\to 2} \overset{\infty}{f(x)} - \overset{5}{h(x)} = \underline{\quad \infty \quad}$

4. $\lim_{x\to 2} \underset{\infty}{g(x)} - \underset{\infty}{f(x)} =$ $\underline{\text{ NEITT }}$ ←

5. $\lim_{x\to 2} \dfrac{f(x)}{g(x)} =$ $\underline{\text{ NEITT }}$ ←

6. $\lim_{x\to 2} \dfrac{h(x) \to 5}{g(x) \to \infty} = \underline{\quad 0 \quad}$

NEITT $\equiv$ Not enough information to tell

intuition Think of

∞ = a really really large positive number

$-\infty$ = a really really large negative number

but realize ∞ and $-\infty$ are relative concepts.

(So for example ① says if you add 2 really large positive numbers the result is another really large positive number.)

Definitions of Inverse Trig Functions from Stewart:

$$\sin^{-1}x = y \iff \sin y = x \quad \text{and} \quad -\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}$$

$$\cos^{-1}x = y \iff \cos y = x \quad \text{and} \quad 0 \leqslant y \leqslant \pi$$

$$\tan^{-1}x = y \iff \tan y = x \quad \text{and} \quad -\frac{\pi}{2} < y < \frac{\pi}{2}$$

most important

$$y = \csc^{-1}x \; (|x| \geqslant 1) \iff \csc y = x \quad \text{and} \quad y \in (0, \pi/2] \cup (\pi, 3\pi/2]$$

$$y = \sec^{-1}x \; (|x| \geqslant 1) \iff \sec y = x \quad \text{and} \quad y \in [0, \pi/2) \cup [\pi, 3\pi/2)$$

$$y = \cot^{-1}x \; (x \in \mathbb{R}) \iff \cot y = x \quad \text{and} \quad y \in (0, \pi)$$

OR These can be written as:

- $\sin(\sin^{-1}x) = x$ for $-1 \leq x \leq 1$
- $\sin^{-1}(\sin x) = x$ for $-\pi/2 \leq x \leq \pi/2$

- $\cos(\cos^{-1}x) = x$ for $-1 \leq x \leq 1$
- $\cos^{-1}(\cos x) = x$ for $0 \leq x \leq \pi$

$\tan(\arctan x) = x$ for any real number x

$\arctan(\tan x) = x$ for $-\pi/2 < x < \pi/2$

- $\sec(\sec^{-1}x) = x$ for $|x| > 1$
- $\sec^{-1}(\sec x) = x$ for $0 \leq x < \pi/2$ or $\pi \leq x < 3\pi/2$

Calculus Properties of Inverse Trig Functions

observe these pairs

- $\frac{d}{dx}\left[\arcsin(x)\right] = \frac{1}{\sqrt{1-x^2}}, \quad -1 < x < 1$

$\frac{d}{dx}\left[\arccos(x)\right] = \frac{-1}{\sqrt{1-x^2}}, \quad -1 < x < 1$

- $\frac{d}{dx}\left[\arctan(x)\right] = \frac{1}{1+x^2}$

$\frac{d}{dx}\left[\cot^{-1}(x)\right] = \frac{-1}{1+x^2}$

most important

$\frac{d}{dx}\left[\operatorname{arcsec}(x)\right] = \frac{1}{x\sqrt{x^2-1}}, \quad |x| > 1$

$\frac{d}{dx}\left[\csc^{-1}(x)\right] = \frac{-1}{x\sqrt{x^2-1}}, \quad |x| > 1$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin(x) + C, \quad -1 \le x \le 1$$

$$\int \frac{-1}{\sqrt{1-x^2}}\,dx = \arccos(x) + C, \quad -1 \le x \le 1$$

$$\int \frac{1}{1+x^2}\,dx = \arctan(x) + C$$

$$\int \frac{1}{x\sqrt{x^2-1}}\,dx = \operatorname{arcsec}(x) + C, \quad |x| > 1$$

Example

To see a relationship between arcsine and arccosine consider the function $f(x)$:

$$f(x) = \arcsin(x) + \arccos(x), \quad -1 \leq x \leq 1$$

$$f'(x) = \frac{1}{\sqrt{1-x^2}} + \frac{-1}{\sqrt{1-x^2}} = 0, \quad -1 < x < 1$$

$\Rightarrow \quad f(x) \cancel{=} \overset{C}{} C$ for some constant C.

Since $f(0) = 0 + \pi/2 = \pi/2$, C must equal $\pi/2$.

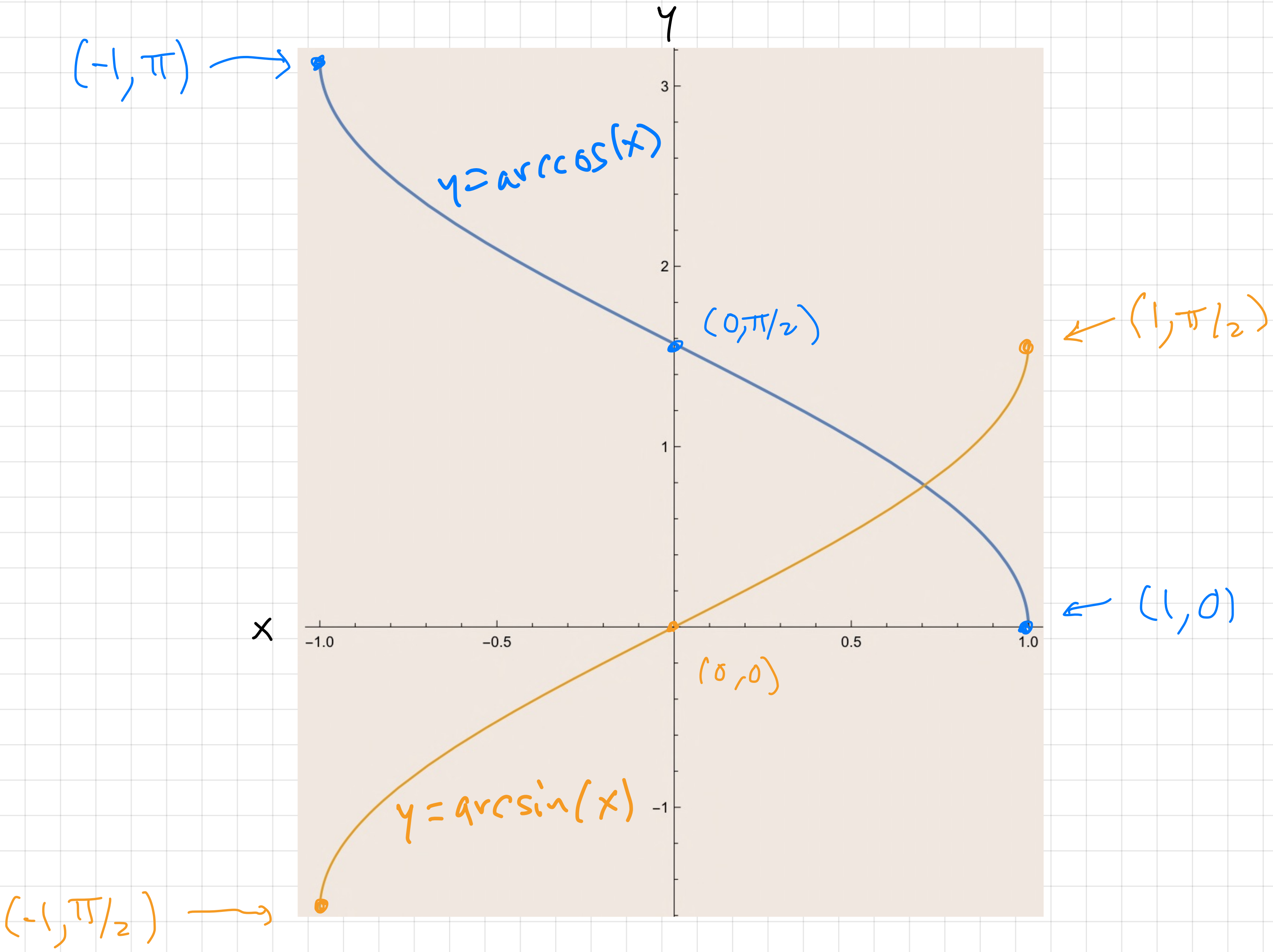
Therefore $\arcsin x + \arccos x = \pi/2$, or

$$\boxed{\arccos(x) = \frac{\pi}{2} - \arcsin(x)}$$

(Is that a formula worth memorizing?
I don't think so....)

$$\arcsin(0) = 0$$
$$\arccos(0) = \pi/2$$

$$\boxed{\arccos(x) = \frac{\pi}{2} - \arcsin(x)}$$



The graph $y = \arccos x$ is obtained from the graph $y = \arcsin x$ by taking its mirror image across the y-axis and then shifting up by $\pi/2$.

Example From Last Class

$$\int \frac{1+x+x^2}{x^3+x}\,dx \overset{*}{=} \int \frac{1}{x} + \frac{1}{1+x^2}\,dx$$

$$= \ln|x| + \arctan(x) + C$$

we'll understand better where this trick came from in Chap 7.

$$* \quad \frac{1+x+x^2}{x^3+x} = \frac{(1+x^2)+x}{(1+x^2)x} =$$

$$= \frac{1+x^2}{(1+x^2)x} + \frac{x}{(1+x^2)x} = \frac{1}{x} + \frac{1}{1+x^2}$$

The functions x^{-p}, $\ln(x)$ and $\arctan(x)$ can be very useful for integrating rational functions $R(x)$. We'll come back to this in Chapter 7.

$$\int \frac{1}{x^p}\,dx = \frac{x^{1-p}}{(1-p)} + C, \quad p \neq 1$$

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int \frac{1}{1+x^2}\,dx = \arctan(x) + C$$

return to

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① $\lim_{x\to 2} f(x) + g(x) =$ ______

② $\lim_{x\to 2} -f(x) =$ ______

③ $\lim_{x\to 2} f(x) - h(x) =$ ______

④ $\lim_{x\to 2} g(x) - f(x) =$ ______

⑤ $\lim_{x\to 2} \dfrac{f(x)}{g(x)} =$ ______

⑥ $\lim_{x\to 2} \dfrac{h(x)}{g(x)} =$ ______

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∞ = a really really large positive number

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but realize ∞ and $-\infty$ are relative concepts.

What does $\lim_{x \to 2} f(x) = \infty$ mean?

- (crude answer)

$\lim_{x \to a} f(x) = \infty$ means that as x gets closer and closer to a, but never equals a, the value of $f(x)$ gets larger and larger without bound.

- (somewhat less vague)

$\lim_{x \to a} f(x) = \infty$ means that for every large positive number N, there is an interval I centered at a so that $f(x) > N$ for every number x in I which does not equal a.

DANGER It's easy to think that $+\infty$ and $-\infty$ are numbers and satisfy all laws of arithmetic — they don't!!

1. $\infty + \infty = \infty$
2. $(-1)\infty = -\infty$
3. $\infty + L = \infty$ where L is a real number
4. $\infty - \infty =$ NEITT
5. $\frac{\infty}{\infty} =$ NEITT

these are called indeterminate forms

6. $\frac{L}{\infty} = 0$ where L is a real number

Some other indeterminate forms are:

$0 \cdot \infty, \ 1^{\infty}, \ \infty^0, \ 0^0, \ 0/0$

There are also some borderline cases that we'll talk about later on.

If we encounter an indeterminate form when calculating the limit of an expression what options are there?

- Use algebra to rewrite the expression
- Use L'Hospital's Rule
- Use a combination of the above

example $\lim_{h\to 0} \frac{\ln(1+h)}{h} = ?$ ← This limit has form 0/0 which is indeterminate

Consider the function $f(x)=\ln(x)$ and find $f'(1)$.

$$f'(x)=\frac{1}{x} \Rightarrow f'(1)=1$$

On the other hand:

because $f(1)=\ln(1)=0$

$$f'(1)=\lim_{h\to 0}\frac{f(1+h)-f(1)}{h} = \lim_{h\to 0}\frac{\ln(1+h)}{h}$$

So $\lim_{h\to 0}\frac{\ln(1+h)}{h} = f'(1) = 1$.

This example suggests that derivatives can be used to calculate some indeterminate limits, and leads to:

L'Hospital's Rule Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a). Suppose that

$$\lim_{x\to a} f(x) = 0 \quad \text{and} \quad \lim_{x\to a} g(x) = 0$$

or that

$$\lim_{x\to a} f(x) = \pm\infty \quad \text{and} \quad \lim_{x\to a} g(x) = \pm\infty$$

(In other words, we have an indeterminate form of type $\frac{0}{0}$ or ∞/∞.) Then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is ∞ or $-\infty$).

example Let $f(x)=\ln(1+x)$, $g(x)=x$

then $\lim_{x\to 0} f(x)=0$ and $\lim_{x\to 0} g(x)=0$

So $\lim_{x\to 0}\frac{\ln(1+x)}{x} \underset{L'H}{=} \lim_{x\to 0}\frac{\frac{1}{1+x}}{1} = \lim_{x\to 0}\frac{1}{1+x} = 1$