Find the values:

- $\tan(\tan^{-1}(1/13)) = 1/13$

Because, by definition,

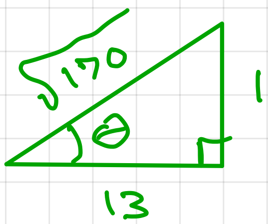
$\tan^{-1}(1/13) =$ Angle θ between $-\pi/2$ and $\pi/2$ with $\tan\theta = 1/13$.

- $\cos(\tan^{-1}(1/13))$

Use right triangle trig:

$\theta = \tan^{-1}(1/13)$

$\tan\theta = 1/13 = \frac{\text{opp}}{\text{adj}}$



$= \cos(\theta) = \frac{\text{adj}}{\text{hyp}}$

$= \frac{13}{\sqrt{170}}$

- $\csc(\tan^{-1}(1/13)) = \frac{\text{hyp}}{\text{opp}} = \sqrt{170}$

- $\sin(\tan^{-1}(1/13)) = \frac{\text{opp}}{\text{hyp}} = \frac{1}{\sqrt{170}}$

- $\sec(\tan^{-1}(1/13)) = \frac{\text{hyp}}{\text{adj}} = \frac{\sqrt{170}}{13}$

BASIC INTEGRATION FORMS:

Stewart p. 543: slightly different list.

E.G. - The "hyperbolic trig functions" $\sinh(x)$, $\cosh(x)$, $\tanh(x)$ are in his list.

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C, \quad n \neq -1$$

$$\int x^{-1}\,dx = \ln|x| + C$$

$$\int e^x\,dx = e^x + C$$

$$\int \ln(x)\,dx = x\ln(x) - x + C$$

$$\int \sin x\,dx = -\cos x + C$$

$$\int \cos x\,dx = \sin x + C$$

$$\int \tan x\,dx = \ln|\sec x| + C = -\ln|\cos x| + C$$

$$\int \sec x\,dx = \ln|\sec x + \tan x| + C$$

$$\int \cot x\,dx = \ln|\sin x| + C$$

$$\int \csc x\,dx = \ln|\csc x - \cot x| + C$$

$$\int \sin^{-1}(x)\,dx = x\sin^{-1}(x) + \sqrt{1-x^2} + C$$

$$\int \cos^{-1}(x)\,dx = x\cos^{-1}(x) - \sqrt{1-x^2} + C$$

$$\int \tan^{-1}(x)\,dx = x\tan^{-1}(x) - \frac{1}{2}\ln(1+x^2) + C$$

$$\int \cos^2(x)\,dx = \frac{1}{2}x + \frac{1}{4}\sin(2x) + C$$

$$\int \sin^2(x)\,dx = \frac{1}{2}x - \frac{1}{4}\sin(2x) + C$$

$$\int \sec^3 x\,dx = \frac{1}{2}\sec x\tan x + \frac{1}{2}\ln|\sec x + \tan x| + C$$

Example: $\displaystyle\int \frac{1}{\sqrt{25-x^2}}\,dx$

---

method 1 : substitution

$$25-x^2 = \frac{1}{\sqrt{25(1-x^2/25)}} = \frac{1}{5}\,\frac{1}{\sqrt{1-(x/5)^2}} \Rightarrow$$

$$\int \frac{1}{\sqrt{25-x^2}}\,dx = \int \frac{1}{\sqrt{1-(x/5)^2}}\,\frac{1}{5}\,dx = \int \frac{1}{\sqrt{1-u^2}}\,du$$

$$\boxed{\begin{aligned} u &= x/5 \\ du &= \tfrac{1}{5}\,dx \end{aligned}}$$

$$= \arcsin(u) + C = \arcsin(x/5) + C$$

---

method 2: "trig substitution"

Put $\underline{x = 5\sin\theta}$, $dx = 5\cos\theta\, d\theta$

$\boxed{\text{really saying: } \sin\theta = x/5, \quad \theta = \arcsin(x/5)}$

$$\Rightarrow \sqrt{25-x^2} = \sqrt{25-25\sin^2\theta} = 5\sqrt{1-\sin^2\theta} = 5\cos\theta$$

$1-\sin^2\theta = \cos^2\theta$

$$\int \frac{1}{\sqrt{25-x^2}}\,dx = \int \frac{1}{5\cos\theta}\,5\cos\theta\,d\theta = \int d\theta$$

$$= \theta + C = 5\arcsin\theta + C$$

$$\boxed{x = 5\sin\theta \Rightarrow \theta = \arcsin(x/5)}$$

This a trig substitution of type 1.

There are 3 types of trig substitution:

Table of Trigonometric Substitutions

	Expression	Substitution	Identity
1	$\sqrt{a^2 - x^2}$	$x = a \sin\theta, \quad -\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$
2	$\sqrt{a^2 + x^2}$	$x = a \tan\theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$
3	$\sqrt{x^2 - a^2}$	$x = a \sec\theta, \quad 0 \leqslant \theta < \frac{\pi}{2}$ or $\pi \leqslant \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$

(Stewart page 526)

$\sqrt{25 - x^2} = \sqrt{a^2 - x^2}$

$a = 5$

For example, here are details for 3.

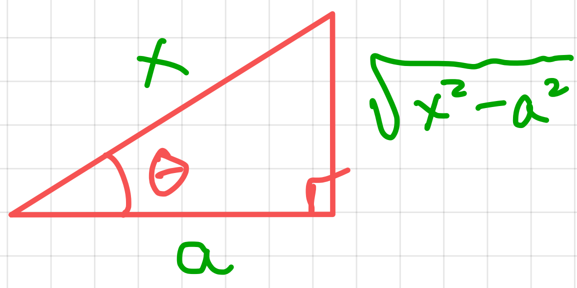
3 For integrals involving $x^2 - a^2$, a = constant

substitute : $\begin{cases} x = a \sec\theta \\ dx = a \sec\theta \tan\theta \, d\theta \end{cases}$

Then $x^2 - a^2 = a^2\sec^2\theta - a^2 = a^2(\sec^2\theta - 1) = a^2\tan^2\theta$

$\sec\theta = \frac{x}{a}$

$= \frac{\text{hyp}}{\text{adj}}$



$\sin(\theta) = \frac{\sqrt{x^2 - a^2}}{x}$

<u>example</u> $\displaystyle\int \frac{1}{x^2-5}\,dx = ?$

Type ③ $a^2 = 5$
$\Rightarrow a = \sqrt{5}$

$$\begin{cases} x = \sqrt{5}\sec\theta \\ dx = \sqrt{5}\sec\theta\tan\theta\,d\theta \end{cases} \qquad x^2-5 = 5\tan^2\theta$$

$$\Rightarrow \frac{1}{x^2-5} = \frac{1}{5}\,\frac{1}{\tan^2\theta}$$

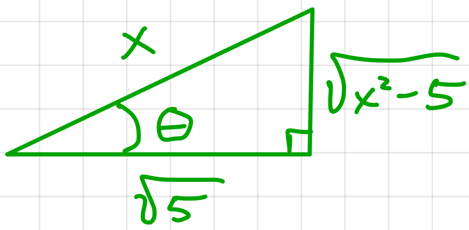
$$= \frac{1/\cancel{\cos\theta}}{\sin\theta/\cancel{\cos\theta}}$$

$$\text{So } \int \frac{1}{x^2-5}\,dx = \int \frac{1}{5}\,\frac{1}{\tan^2\theta}\sqrt{5}\sec\theta\tan\theta\,d\theta$$

$$= \frac{1}{\sqrt{5}}\int \frac{\sec\theta}{\tan\theta}\,d\theta = \frac{1}{\sqrt{5}}\int \frac{1}{\sin\theta}\,d\theta$$

$$= \frac{1}{\sqrt{5}}\int \csc(\theta)\,d\theta$$

$$= \frac{1}{\sqrt{5}}\ln|\csc\theta - \cot\theta| + C$$



$$\sec\theta = \frac{x}{\sqrt{5}} = \frac{\text{hyp}}{\text{adj}}$$

$$\csc\theta = \frac{\text{hyp}}{\text{opp}} = \frac{x}{\sqrt{x^2-5}}$$

$$\cot\theta = \frac{\text{adj}}{\text{opp}} = \frac{\sqrt{5}}{\sqrt{x^2-5}}$$

$$= \frac{1}{\sqrt{5}}\ln\left|\frac{x}{\sqrt{x^2-5}} - \frac{\sqrt{5}}{\sqrt{x^2-5}}\right| + C$$

$$= \frac{1}{\sqrt{5}}\ln\left|\frac{x-\sqrt{5}}{\sqrt{x^2-5}}\right| + C$$

$$= \frac{1}{\sqrt{5}}\left(\ln|x-\sqrt{5}| - \ln\sqrt{x^2-5}\right) + C$$

Example

$$\int \frac{1}{(x^2+1)^2}\,dx$$

$$\begin{cases} x = \tan\theta \\ x^2+1 = \sec^2\theta \\ dx = \sec^2\theta\, d\theta \end{cases} \quad \Leftarrow \theta = \arctan(x)$$

$\sqrt{x^2+a^2}$, $a = 1$

$$\int \frac{1}{(x^2+1)^2}\,dx = \int \frac{1}{\sec^4\theta}\sec^2\theta\, d\theta$$

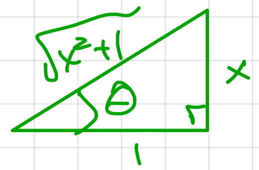
$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$

$$= \int \frac{1}{\sec^2\theta}\,d\theta = \int \cos^2\theta\, d\theta$$

$\boxed{\sin(2\theta) = 2\sin\theta\cos\theta}$

$$= \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta + C$$

$$= \frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta + C$$



$\tan\theta = \frac{x}{1}$

$\sin\theta = x/\sqrt{x^2+1}$

$\cos\theta = 1/\sqrt{x^2+1}$

$$= \frac{1}{2}\arctan(x) + \frac{1}{2}\frac{x}{\sqrt{x^2+1}}\frac{1}{\sqrt{x^2+1}} + C$$

$$= \frac{1}{2}\arctan(x) + \frac{1}{2}\frac{x}{x^2+1} + C$$

NOTE: The integrand in this example is a rational function:

$$\frac{1}{(x^2+1)^2} = \frac{\text{degree 0 polynomial}}{\text{degree 4 polynomial}}$$

In fact its a very special type of rational function called a "partial fraction".

Technique of Trig Substitution — Recap

① For integrals involving $a^2 - x^2$, $a = \text{constant}$

substitute : $\begin{cases} x = a\sin\theta \\ dx = a\cos\theta\, d\theta \end{cases}$

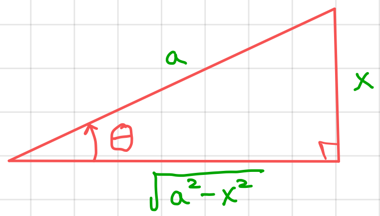
Then $a^2 - x^2 = a^2 - a^2\sin^2\theta = a^2(1 - \sin^2\theta) = a^2\cos^2\theta$

$\sin\theta = \dfrac{x}{a} = \dfrac{\text{opp}}{\text{hyp}}$

$\Rightarrow$

$\cos(\theta) = \dfrac{\sqrt{a^2 - x^2}}{a}$

$\tan(\theta) = \dfrac{x}{\sqrt{a^2 - x^2}}$



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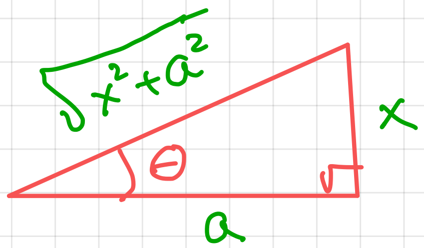
② For integrals involving $x^2 + a^2$, $a = \text{constant}$

substitute : $\begin{cases} x = a\tan\theta \\ dx = a\sec^2\theta\, d\theta \end{cases}$

Then $a^2 + x^2 = a^2 + a^2\tan^2\theta = a^2(1 + \tan^2\theta) = a^2\sec^2\theta$

$\tan\theta = \dfrac{x}{a}$



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③ For integrals involving $x^2 - a^2$, $a = \text{constant}$

substitute : $\begin{cases} x = a\sec\theta \\ dx = a\sec\theta\tan\theta\, d\theta \end{cases}$

Then $x^2 - a^2 = a^2\sec^2\theta - a^2 = a^2(\sec^2\theta - 1) = a^2\tan^2\theta$

$\sec\theta = \dfrac{x}{a}$

$= \dfrac{\text{hyp}}{\text{adj}}$