$\boxed{A=-2,\ B=5}$

Problem

rational function

Find numbers A and B for which:

$$\frac{3-2x}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2}$$

method 1

$$\frac{A}{x+1} + \frac{B}{(x+1)^2} = \frac{A}{x+1}\frac{x+1}{x+1} + \frac{B}{(x+1)^2}$$

$$= \frac{Ax + (A+B)}{(x+1)^2} = \frac{3-2x}{(x+1)^2}$$

So $A=-2$, $A+B=3 \Rightarrow B = 3-A = 3-(-2)$
$= 5$

method 2

Plug in $x=0$: $\frac{3-2(0)}{(0+1)^2} = 3 = \frac{A}{0+1} + \frac{B}{0+1}$

$\Rightarrow A+B=3$

Plug in $x=1$: $\left(\frac{1}{4} = \frac{A}{2} + \frac{B}{4}\right)4$

$$\begin{cases} 2A+B=1 \\ A+B=3 \end{cases}$$

$A=-2$ $\quad$ $B=3-A=5$

from Exam 3 Take-Home

Problem 3(d):

sin²x cos²x is constant with respect to θ

$$\int \sin^2(x)\cos^2(x)\, d\theta = \sin^2(x)\cos^2(x)\,\theta + C$$

$$\int \sin^2(x)\cos^2(x)\, dx =$$

$\sin^2(x) = \frac{1}{2}(1-\cos 2x)$,
$\cos^2(x) = \frac{1}{2}(1+\cos 2x)$,
etc.

PROBLEM 4. A student determines that

$$\int \frac{2x^2}{\sqrt{1-x^2}}\, dx = -x\sqrt{1-x^2} + \arcsin(x) + C.$$

Is that answer correct? Explain.

We can check by differentiating the answer:

$$\frac{d}{dx}\left[\left(-x(1-x^2)^{1/2}\right) + \arcsin(x)\right] =$$

use product rule...

$$\left(-(1-x^2)^{1/2} - x\,\tfrac{1}{2}(1-x^2)^{-1/2}(-2x)\right) + \frac{1}{\sqrt{1-x^2}}$$

$$= -\sqrt{1-x^2} + \frac{x^2}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-x^2}}$$

$$= \frac{-\sqrt{1-x^2}\sqrt{1-x^2}}{\sqrt{1-x^2}} + \frac{x^2}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-x^2}}$$

$$= \frac{-(1-x^2) + x^2 + 1}{\sqrt{1-x^2}} = \frac{2x^2}{\sqrt{1-x^2}}$$

So the answer is correct.

Basic Integration Forms:

Stewart p. 543: slightly different list.

E.G. - The "hyperbolic trig functions" $\sinh(x)$, $\cosh(x)$, $\tanh(x)$ are in his list.

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C,\ n \neq -1$$

$$\int x^{-1}\,dx = \ln|x| + C$$

$$\int e^x\,dx = e^x + C$$

$$\int \ln(x)\,dx = x\ln(x) - x + C$$

$$\int \sin x\,dx = -\cos x + C$$

$$\int \cos x\,dx = \sin x + C$$

$$\int \tan x\,dx = \ln|\sec x| + C = -\ln|\cos x| + C$$

$$\int \sec x\,dx = \ln|\sec x + \tan x| + C$$

$$\int \cot x\,dx = \ln|\sin x| + C$$

$$\int \csc x\,dx = \ln|\csc x - \cot x| + C$$

$$\int \sin^{-1}(x)\,dx = x\sin^{-1}(x) + \sqrt{1-x^2} + C$$

$$\int \cos^{-1}(x)\,dx = x\cos^{-1}(x) - \sqrt{1-x^2} + C$$

$$\int \tan^{-1}(x)\,dx = x\tan^{-1}(x) - \frac{1}{2}\ln(1+x^2) + C$$

$$\int \cos^2(x)\,dx = \frac{1}{2}x + \frac{1}{4}\sin(2x) + C$$

$$\int \sin^2(x)\,dx = \frac{1}{2}x - \frac{1}{4}\sin(2x) + C$$

$$\int \sec^3 x\,dx = \frac{1}{2}\sec x\tan x + \frac{1}{2}\ln|\sec x + \tan x| + C$$

From Class Notes 4/16 (Stewart section 7.5)

7.5 EXERCISES

1–82 Evaluate the integral.

1. $\int \frac{\cos x}{1-\sin x}\,dx$
2. $\int_0^1 (3x+1)^{\sqrt{2}}\,dx$
3. $\int_1^4 \sqrt{y}\ln y\,dy$
4. $\int \frac{\sin^3 x}{\cos x}\,dx$
5. $\int \frac{t}{t^4+2}\,dt$
6. ~~6.~~ $\int_0^1 \frac{x}{(2x+1)^3}\,dx$
7. $\int_{-1}^1 \frac{e^{\arctan y}}{1+y^2}\,dy$
8. $\int t\sin t\cos t\,dt$
9. ~~9.~~ $\int_2^4 \frac{x+2}{x^2+3x-4}\,dx$
10. $\int \frac{\cos(1/x)}{x^3}\,dx$
11. ~~11.~~ $\int \frac{1}{x^3\sqrt{x^2-1}}\,dx$
12. ~~12.~~ $\int \frac{2x-3}{x^3+3x}\,dx$
13. $\int \sin^5 t\cos^4 t\,dt$
14. ~~14.~~ $\int \ln(1+x^2)\,dx$
15. $\int x\sec x\tan x\,dx$
16. ~~16.~~ $\int_0^{\sqrt{2}/2} \frac{x^2}{\sqrt{1-x^2}}\,dx$
17. $\int_0^\pi t\cos^2 t\,dt$
18. $\int_1^4 \frac{e^{\sqrt{t}}}{\sqrt{t}}\,dt$
19. $\int e^{x+e^x}\,dx$
20. $\int e^2\,dx = e^2x + C$
21. ~~21.~~ $\int \arctan\sqrt{x}\,dx$
22. $\int \frac{\ln x}{x\sqrt{1+(\ln x)^2}}\,dx$

Some Hints

#1: multiply by $(1+\sin x)/(1+\sin x)$. (see next page)

#2: $\int x^p\,dx = \frac{1}{p+1}x^{p+1} + C$. Take $p=\sqrt{2}$.

#3: Try IBP with $u = \ln(y)$

#4: $\frac{\sin^3(x)}{\cos(x)} = \frac{1-\cos^2(x)}{\cos(x)}\cdot\sin(x)$

#5: substitute $u=t^2$ and observe $t^4=u^2$

#7: $\frac{d}{dy}\left[\arctan(y)\right] = ?$

#8: IBP

#10: $u = 1/x$ (see next page)

#13: $\sin^5 t = (1-\cos^2 t)^2\sin t$

#17: Try IBP with $u = t\cos t$

#19: $e^{x+e^x} = e^{e^x}e^x$ take $u=e^x$

#20: easiest of all

#22: substitute $u = 1+\ln(x)^2$

We can now work some of these problems that we couldn't work before.

$\theta = \sec^{-1}(x)$

#11 from section 7.5

form $\sqrt{x^2 - a^2}$

$a = 1$

$$\int \frac{1}{x^3\sqrt{x^2-1}}\,dx$$

$$\begin{cases} x = \sec\theta \\ dx = \sec\theta\tan\theta\,d\theta \\ x^2 - 1 = \tan^2\theta \end{cases}$$

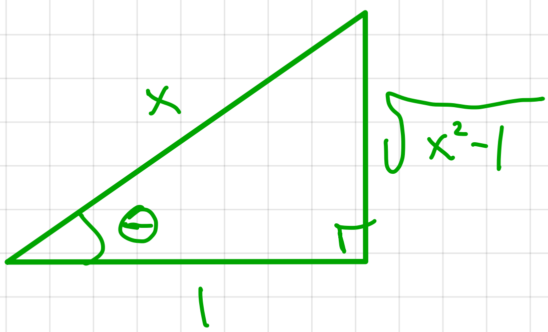
$$= \int \frac{1}{(\sec\theta)^{\cancel{3}\,2}\cdot\cancel{\tan\theta}}\cdot\cancel{\sec\theta}\,\cancel{\tan\theta}\,d\theta$$

$$= \int \frac{1}{\sec^2\theta}\,d\theta = \int \cos^2\theta\,d\theta$$

$$= \int \frac{1}{2}(1+\cos 2\theta)\,d\theta = \frac{\theta}{2} + \frac{1}{4}\sin 2\theta + C$$

$$= \frac{\theta}{2} + \frac{1}{2}\sin\theta\cos\theta + C$$

$$= \frac{1}{2}\sec^{-1}x + \frac{1}{2}\,\frac{\sqrt{x^2-1}}{x}\,\frac{1}{x} + C$$



$\sec\theta = \frac{x}{1} = \frac{\text{hyp}}{\text{adj}}$

$\sin(2\theta) = 2\sin\theta\cos\theta$

$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{\sqrt{x^2-1}}{x}$

$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{1}{x}$

There are 3 types of trig substitution:

Table of Trigonometric Substitutions

	Expression	Substitution	Identity
1	$\sqrt{a^2 - x^2}$	$x = a \sin\theta, \quad -\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$
2	$\sqrt{a^2 + x^2}$	$x = a \tan\theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$
3	$\sqrt{x^2 - a^2}$	$x = a \sec\theta, \quad 0 \leqslant \theta < \frac{\pi}{2}$ or $\pi \leqslant \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$

(Stewart page 526, Section 7.3)

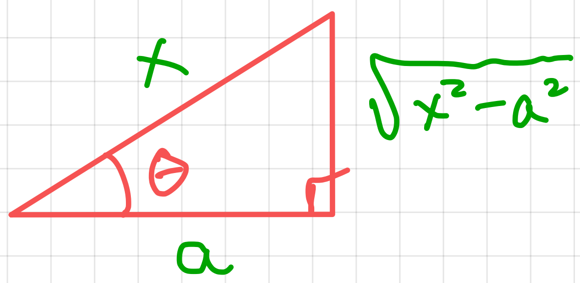
For example, here are details for 3.

3 For integrals involving $x^2 - a^2$, a = constant

substitute : $\begin{cases} x = a \sec\theta \\ dx = a \sec\theta \tan\theta \, d\theta \end{cases}$

Then $x^2 - a^2 = a^2 \sec^2\theta - a^2 = a^2(\sec^2\theta - 1) = a^2 \tan^2\theta$

$\sec\theta = \frac{x}{a} = \frac{\text{hyp}}{\text{adj}}$



Integrating Rational Functions

section 7.4

There is a procedure through which any rational function can be integrated (in theory). Three ingredients:

① There is a family of special rational functions known as "partial fractions".

② Every rational function $P(x)/Q(x)$ with $\text{degree}(P) < \text{degree}(Q)$ can be written as a sum of partial fractions.

③ Previous integration techniques can be used to integrate any partial fraction.

$$\text{rational function} = \frac{P(x)}{Q(x)} = \frac{\text{polynomial}}{\text{polynomial}}$$

Note: For technical reasons we focus on rational functions $P(x)/Q(x)$ where $\text{degree}(P) < \text{degree}(Q)$. But we'll give some examples later that describe what to do when $\text{degree}(P) \geq \text{degree}(Q)$.

<u>example</u> (from start of class)

$$f(x) = \frac{3-x}{x^2+2x+1}$$

$$f(x) = \frac{3-2x}{(x+1)^2} = \frac{P(x)}{Q(x)}$$

is a rational function where $P(x) = 3-2x$, $Q(x) = (x+1)^2$

(here degree$(P)=1$, degree$(Q)=2$)

We can write

$$f(x) = \frac{-2}{x+1} + \frac{5}{(x+1)^2}$$

where each of $\frac{-2}{x+1}$ and $\frac{5}{(x+1)^2}$ are "partial fractions", but $f(x)$ itself is not.

$$\int \frac{-2}{x+1}\,dx = -2\ln|x+1| + C \qquad \begin{cases} u = x+1 \\ du = dx \end{cases}$$

$$\int \frac{5}{(x+1)^2}\,dx = -5\frac{1}{x+1} + C$$

$$\Rightarrow \int f(x)\,dx = -2\ln|x+1| - \frac{5}{x+1} + C$$

So in practice there are three points to discuss:

① What is a partial fraction?

② Given $P(x)/Q(x)$ with $\text{degree}(P) < \text{degree}(Q)$ how can we write it as a sum of partial fractions?

③ How do you integrate a partial fraction?

Comment: The hardest point is ②!

Technique of Trig Substitution

① For integrals involving a^2-x^2, $a=\text{constant}$

substitute: $\begin{cases} x = a\sin\theta \\ dx = a\cos\theta\, d\theta \end{cases}$

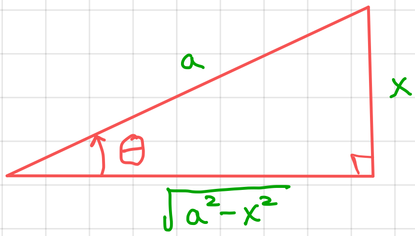
Then $a^2-x^2 = a^2 - a^2\sin^2\theta = a^2(1-\sin^2\theta) = a^2\cos^2\theta$

$\sin\theta = \dfrac{x}{a} = \dfrac{\text{opp}}{\text{hyp}}$

$\Rightarrow$

$\cos(\theta) = \dfrac{\sqrt{a^2-x^2}}{a}$

$\tan(\theta) = \dfrac{x}{\sqrt{a^2-x^2}}$



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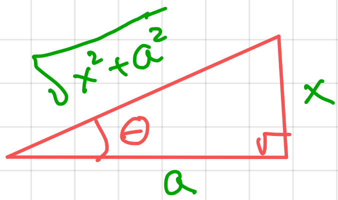
② For integrals involving x^2+a^2, $a=\text{constant}$

substitute: $\begin{cases} x = a\tan\theta \\ dx = a\sec^2\theta\, d\theta \end{cases}$

Then $a^2+x^2 = a^2+a^2\tan^2\theta = a^2(1+\tan^2\theta) = a^2\sec^2\theta$

$\tan\theta = \dfrac{x}{a}$



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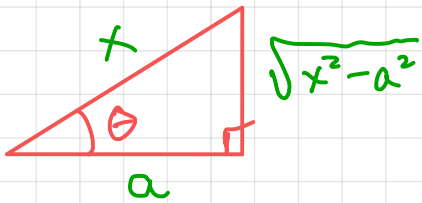
③ For integrals involving x^2-a^2, $a=\text{constant}$

substitute: $\begin{cases} x = a\sec\theta \\ dx = a\sec\theta\tan\theta\, d\theta \end{cases}$

Then $x^2-a^2 = a^2\sec^2\theta - a^2 = a^2(\sec^2\theta-1) = a^2\tan^2\theta$

$\sec\theta = \dfrac{x}{a}$

$= \dfrac{\text{hyp}}{\text{adj}}$



example $\int_0^7 \sqrt{49-x^2}\,dx = ?$

$$\begin{cases} x = 7\sin\theta \\ dx = 7\cos\theta\,d\theta \\ 49 - x^2 = 49\cos^2\theta \end{cases}$$