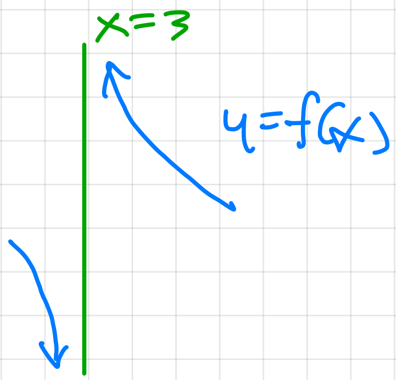
Questions:

(1) The graph $y=f(x)$ of the rational function $f(x) = \dfrac{x-2}{x^2-2x-3}$ has two vertical asymptotes where are they? $x^2-2x-3 = (x-3)(x+1)$

So $x=3$ and $x=-1$ are vertical asymptotes.

$$\left.\begin{array}{l} \lim\limits_{\substack{x\to 3+ \\ x>3}} f(x) = \infty \\ \lim\limits_{\substack{x\to 3- \\ x<3}} f(x) = -\infty \end{array}\right\} \Rightarrow$$



(2) Does the graph of every rational function have a vertical asymptote? NO e.g. $f(x) = \dfrac{1}{x^2+49}$

(3) When does the graph of a rational function $f(x) = \dfrac{P(x)}{Q(x)}$ have a horizontal asymptote?

ANS: Whenever $\text{degree}(P) \le \text{degree}(Q)$.

Moreover, the x-axis is a horizontal asymptote whenever $\text{degree}(P) < \text{degree}(Q)$

The rational function $P(x)/Q(x)$ is called proper when this happens.

Technique of Trig Substitution ⟵ (review)

① For integrals involving $a^2 - x^2$, $a = \text{constant}$

substitute : $\begin{cases} x = a\sin\theta \\ dx = a\cos\theta\, d\theta \end{cases}$

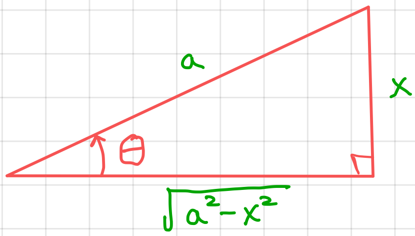
Then $a^2 - x^2 = a^2 - a^2\sin^2\theta = a^2(1-\sin^2\theta) = a^2\cos^2\theta$

$\sin\theta = \dfrac{x}{a} = \dfrac{\text{opp}}{\text{hyp}}$

$\Rightarrow$

$\cos(\theta) = \dfrac{\sqrt{a^2-x^2}}{a}$

$\tan(\theta) = \dfrac{x}{\sqrt{a^2-x^2}}$



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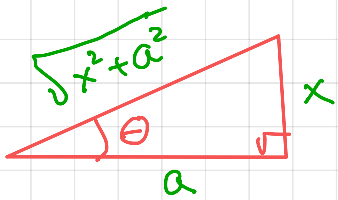
② For integrals involving $x^2 + a^2$, $a = \text{constant}$

substitute : $\begin{cases} x = a\tan\theta \\ dx = a\sec^2\theta\, d\theta \end{cases}$

Then $a^2 + x^2 = a^2 + a^2\tan^2\theta = a^2(1+\tan^2\theta) = a^2\sec^2\theta$

$\tan\theta = \dfrac{x}{a}$



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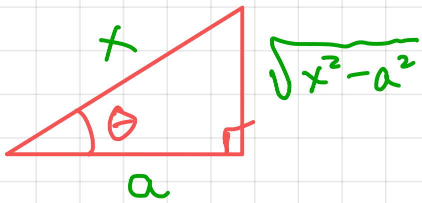
③ For integrals involving $x^2 - a^2$, $a = \text{constant}$

substitute : $\begin{cases} x = a\sec\theta \\ dx = a\sec\theta\tan\theta\, d\theta \end{cases}$

Then $x^2 - a^2 = a^2\sec^2\theta - a^2 = a^2(\sec^2\theta - 1) = a^2\tan^2\theta$

$\sec\theta = \dfrac{x}{a}$

$= \dfrac{\text{hyp}}{\text{adj}}$



example $\int_0^7 \sqrt{49-x^2}\,dx = ?$ $\leftarrow \sqrt{a^2-x^2}$ form, $a=7$

$$\begin{cases} x = 7\sin\theta \\ dx = 7\cos\theta\,d\theta \\ 49-x^2 = 49\cos^2\theta \end{cases} \Rightarrow \begin{cases} \theta = \sin^{-1}(x/7) \\ \theta(0) = \sin^{-1}(0) = 0 \\ \theta(7) = \sin^{-1}(1) = \pi/2 \end{cases}$$

$$\int_0^7 \sqrt{49-x^2}\,dx = \int_0^{\pi/2} \sqrt{49\cos^2\theta} \cdot 7\cos\theta\,d\theta$$

$$= 49\int_0^{\pi/2} \cos^2\theta\,d\theta$$

$$= 49\int_0^{\pi/2} \frac{1}{2}(1+\cos 2\theta)\,d\theta$$

$$= \frac{49}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right)\Big|_{\theta=0}^{\pi/2}$$

$$= \frac{49}{2}\left(\left(\frac{\pi}{2} + \frac{1}{2}\sin(\pi)\right) - \left(0 + \frac{1}{2}\sin(0)\right)\right)$$

$$= 49\pi/4$$

Can this integral be interpreted as an area?

$$R: \begin{cases} 0 \le x \le 7 \\ 0 \le y \le \sqrt{49-x^2} \end{cases} \leftarrow$$ Yes because $\sqrt{49-x^2} \ge 0$ for $0 \le x \le 7$.

$$\text{Area}(R) = \int_0^7 \sqrt{49-x^2}\,dx = \frac{49\pi}{4}$$

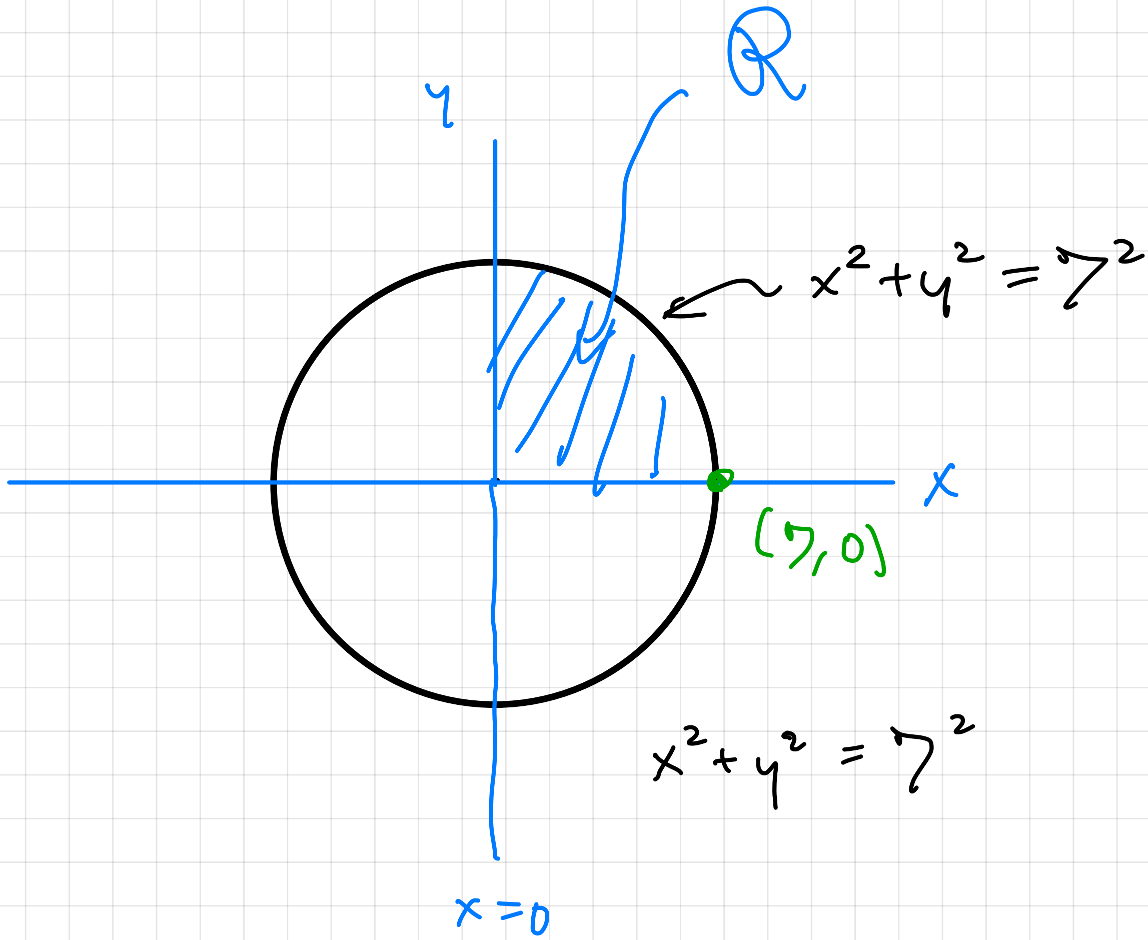
Can this integral be interpreted as an area? **YES**

$$R: \begin{cases} 0 \le x \le 7 \\ 0 \le y \le \sqrt{49-x^2} \end{cases}$$

$$\Rightarrow \quad \text{Area}(R) = \int_0^7 \sqrt{49-x^2}\, dx$$

Draw the picture??

$$y = \sqrt{49-x^2} \Rightarrow y^2 = 49 - x^2 \Rightarrow x^2 + y^2 = 7^2$$



$$\underset{\text{of circle}}{\text{area}} = \pi(7)^2 = 49\pi$$

R = Quarter sector of circle with area $49\pi/4$.

Example (done last week using $x = \tan\theta$)

$$\int \frac{1}{(x^2+1)^2}\,dx = \frac{1}{2}\arctan(x) + \frac{1}{2}\,\frac{x}{x^2+1} + C$$

form $\sqrt{x^2+a^2}$, $a=1$ $x = \tan\theta$

Example

$$\int \frac{1}{(2x^2+2)^2}\,dx = \frac{1}{4}\int \frac{1}{(x^2+1)^2}\,dx$$

$\frac{1}{2^2(x^2+1)^2}$ "

$$= \frac{1}{8}\arctan(x) + \frac{1}{8}\,\frac{x}{x^2+1} + C$$

Example

$$\frac{1}{2}\int \frac{x}{(x^2+1)^2}\,2\,dx = \,?$$

← could use $x=\tan\theta$ but,

Here's its easier to use a u-substitution

$$\begin{cases} u = x^2+1 \\ du = 2x\,dx \end{cases}$$

$$\int \frac{x}{(x^2+1)^2}\,dx = \frac{1}{2}\int \frac{1}{u^2}\,du = -\frac{1}{2}\,\frac{1}{u} + C$$

$$= -\frac{1}{2}\,\frac{1}{x^2+1} + C$$

NOTE: All 3 of these integrands are "partial fractions" which are special types of rational functions.

rational function $= \dfrac{P(x)}{Q(x)} = \dfrac{\text{polynomial in } x}{\text{polynomial in } x}$

A rational function $P(x)/Q(x)$ is <u>proper</u> when Degree $P(x)$ < degree $Q(x)$.

Integrating Rational Functions (section 7.4)

There is a procedure through which any rational function can be integrated (in theory). Three ingredients:

① There is a family of special rational functions known as "partial fractions".

② Every proper rational function can be expressed as a sum of partial fractions.

③ Previous integration techniques can be used to integrate any partial fraction.

<u>NOTE:</u> There is a simple procedure to deal with rational functions which are not proper. We'll give some examples later to illustrate the procedure.

So in practice there are three points to discuss:

① What is a partial fraction?

② Given $P(x)/Q(x)$ with $\mathrm{degree}(P) < \mathrm{degree}(Q)$ how can we write it as a sum of partial fractions?

③ How do you integrate a partial fraction?

Comment: The hardest point is ②!

Before saying what a partial fraction is its important to review some basic information about quadratic polynomials.

A quadratic polynomial has the form

$$P(x) = ax^2 + bx + c$$

Some quadratics can be factored as a product

$$ax^2 + bx + c = (a_1 x + b_1)(a_2 x + b_2)$$

of two linear terms. When does this happen?

answer: It happens when the equation $ax^2+bx+c=0$
* has a (real number) solution. The quadratic formula says the roots of $ax^2+bx+c=0$ are

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

so ax^2+bx^2+c can be factored into two linear terms whenever the discriminant b^2-4ac is ≥ 0.

When $b^2-4ac<0$ we say that the quadratic ax^2+bx+c is irreducible.

* If ax^2+bx+c factors as $(a_1x+b_1)(a_2x+b_2)$ then $x=-b_1/a_1$ and $x=-b_2/a_2$ are solutions to the equation $ax^2+bx+c=0$. (But note that $-b_1/a_1$ and $-b_2/a_2$ will be 'repeated' roots if $-b_1/a_1=-b_2/a_2$.)

Some examples

- x^2-2x-3 is "reducible".

It's discriminant is $(-2)^2-4(1)(-3)=16$

and $x^2-2x-3 = (x+1)(x-3)$

$\leftarrow b^2-4ac = 0^2-4(1)(49)$
$= -196 <$

- x^2+49 is irreducible.

- $4x^2+12x+9$ is reducible.

It's discriminant is $0=(12)^2-4(4)(9)$

and $4x^2+12x+9 = (2x+3)(2x+3)$

is a perfect square.

- x^2-2x+3 is irreducible

discriminant $=(-2)^2-4(1)(3) = -8$

A **partial fraction** is a rational function having one of the forms:

$$\frac{A}{(ax+b)^i} \quad \leftarrow \text{"linear form"}$$

or

$$\frac{Ax+B}{(ax^2+bx+c)^i} \quad \leftarrow \text{"irreducible quadratic form"}$$

where A, B, a, b, c, i are constants for which i is a positive integer and ax^2+bx+c is an irreducible quadratic.

Theorem Every proper rational function $f(x)$ can be expressed in one and only one way as a sum of partial fractions.

example

$\dfrac{3-2x}{(x+1)^2}$ is not a partial fraction but

$$\frac{3-2x}{(x+1)^2} = \frac{-2}{x+1} + \frac{5}{(x+1)^2}$$ where

$-\dfrac{2}{x+1}$ and $\dfrac{5}{(x+1)^2}$ are partial fractions.

example

$\dfrac{3-2x}{(x^2+1)^2}$ is a partial fraction.