① Express $\sin(4x)$ in terms of powers of $\sin(x)$ and $\cos(x)$.

recall:

$$\boxed{\begin{aligned} \sin(2x) &= 2\sin(x)\cos(x) \\ \cos(2x) &= \cos^2(x) - \sin^2(x) \\ &= 1 - 2\sin^2(x) \end{aligned}}$$

$$\begin{aligned} \sin(4x) &= 2\sin(2x)\cos(2x) \\ &= 2\left(2\sin(x)\cos(x)\right)\left(1-2\sin^2(x)\right) \\ &= 4\sin(x)\cos(x) - 8\sin^3(x)\cos(x) \end{aligned}$$

② $\frac{1}{4}\int \sin(4x)\,4\,dx = ?$ ← hint: don't use ①

$$\begin{cases} u = 4x \\ du = 4\,dx \end{cases}$$

$$\begin{aligned} &= \frac{1}{4}\int \sin(u)\,du = -\frac{1}{4}\cos u + C \\ &= -\frac{1}{4}\cos(4x) + C \end{aligned}$$

③ $\int \sin(x)\sin(4x)\,dx = ?$ ← hint: use ①

$$\begin{aligned} &= \int \sin(x)\left(4\sin(x)\cos(x) - 8\sin^3(x)\cos(x)\right)dx \\ &= \int 4\sin^2(x)\cos(x) - 8\sin^4(x)\cos(x)\,dx \\ &= \int \left(4\sin^2(x) - 8\sin^4(x)\right)\cos(x)\,dx \end{aligned}$$

$$\begin{cases} u = \sin(x) \\ du = \cos(x)\,dx \end{cases}$$

$$\begin{aligned} &= \int 4u^2 - 8u^4\,du \\ &= \frac{4}{3}u^3 - \frac{8}{5}u^5 + C \\ &= \frac{4}{3}\sin^3(x) - \frac{8}{5}\sin^5(x) + C \end{aligned}$$

7.5 EXERCISES

Stewart - page 523

1–82 Evaluate the integral.

1. $\displaystyle\int \frac{\cos x}{1 - \sin x}\,dx$

2. $\displaystyle\int_0^1 (3x + 1)^{\sqrt{2}}\,dx$

3. $\displaystyle\int_1^4 \sqrt{y}\,\ln y\,dy$

4. $\displaystyle\int \frac{\sin^3 x}{\cos x}\,dx$

5. $\displaystyle\int \frac{t}{t^4 + 2}\,dt$

6. $\displaystyle\int_0^1 \frac{x}{(2x + 1)^3}\,dx$

7. $\displaystyle\int_{-1}^1 \frac{e^{\arctan y}}{1 + y^2}\,dy$

8. $\displaystyle\int t \sin t \cos t\,dt$

9. $\displaystyle\int_2^4 \frac{x + 2}{x^2 + 3x - 4}\,dx$

10. $\displaystyle\int \frac{\cos(1/x)}{x^3}\,dx$

11. $\displaystyle\int \frac{1}{x^3\sqrt{x^2 - 1}}\,dx$

12. $\displaystyle\int \frac{2x - 3}{x^3 + 3x}\,dx$

13. $\displaystyle\int \sin^5 t\,\cos^4 t\,dt$

14. $\displaystyle\int \ln(1 + x^2)\,dx$

15. $\displaystyle\int x \sec x \tan x\,dx$

16. $\displaystyle\int_0^{\sqrt{2}/2} \frac{x^2}{\sqrt{1 - x^2}}\,dx$

17. $\displaystyle\int_0^\pi t\cos^2 t\,dt$

18. $\displaystyle\int_1^4 \frac{e^{\sqrt{t}}}{\sqrt{t}}\,dt$

19. $\displaystyle\int e^{x + e^x}\,dx$

20. $\displaystyle\int e^2\,dx$

21. $\displaystyle\int \arctan\sqrt{x}\,dx$

22. $\displaystyle\int \frac{\ln x}{x\sqrt{1 + (\ln x)^2}}\,dx$

23. $\displaystyle\int_0^1 (1 + \sqrt{x})^8\,dx$

24. $\displaystyle\int (1 + \tan x)^2 \sec x\,dx$

25. $\displaystyle\int_0^1 \frac{1 + 12t}{1 + 3t}\,dt$

26. $\displaystyle\int_0^1 \frac{3x^2 + 1}{x^3 + x^2 + x + 1}\,dx$

27. $\displaystyle\int \frac{dx}{1 + e^x}$

28. $\displaystyle\int \sin\sqrt{at}\,dt$

29. $\displaystyle\int \ln\left(x + \sqrt{x^2 - 1}\right)dx$

30. $\displaystyle\int_{-1}^2 |\,e^x - 1\,|\,dx$

31. $\displaystyle\int \sqrt{\frac{1 + x}{1 - x}}\,dx$

32. $\displaystyle\int_1^3 \frac{e^{3/x}}{x^2}\,dx$

33. $\displaystyle\int \sqrt{3 - 2x - x^2}\,dx$

34. $\displaystyle\int_{\pi/4}^{\pi/2} \frac{1 + 4\cot x}{4 - \cot x}\,dx$

35. $\displaystyle\int_{-\pi/2}^{\pi/2} \frac{x}{1 + \cos^2 x}\,dx$

36. $\displaystyle\int \frac{1 + \sin x}{1 + \cos x}\,dx$

37. $\displaystyle\int_0^{\pi/4} \tan^3\theta\,\sec^2\theta\,d\theta$

38. $\displaystyle\int_{\pi/6}^{\pi/3} \frac{\sin\theta\,\cot\theta}{\sec\theta}\,d\theta$

39. $\displaystyle\int \frac{\sec\theta\,\tan\theta}{\sec^2\theta - \sec\theta}\,d\theta$

40. $\displaystyle\int_0^\pi \sin 6x \cos 3x\,dx$

41. $\displaystyle\int \theta\tan^2\theta\,d\theta$

42. $\displaystyle\int \frac{\tan^{-1}x}{x^2}\,dx$

43. $\displaystyle\int \frac{\sqrt{x}}{1 + x^3}\,dx$

44. $\displaystyle\int \sqrt{1 + e^x}\,dx$

45. $\displaystyle\int x^5 e^{-x^3}\,dx$

46. $\displaystyle\int \frac{(x - 1)e^x}{x^2}\,dx$

47. $\displaystyle\int x^3(x - 1)^{-4}\,dx$

48. $\displaystyle\int_0^1 x\sqrt{2 - \sqrt{1 - x^2}}\,dx$

Color Key:

≡ See 4/14 class notes

≡ rational functions (only #25 is not proper)

≡ trig substitution

≡ rework to get a rational function.

A quadratic polynomial has the form

$$P(x) = ax^2 + bx + c$$

Some quadratics can be factored as a product

$$ax^2 + bx + c = (a_1 x + b_1)(a_2 x + b_2)$$

of two linear terms. When does this happen?

answer: It happens when the equation $ax^2+bx+c=0$ ⊛ has a (real number) solution. The quadratic formula says the roots of $ax^2+bx+c=0$ are

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

so ax^2+bx^2+c can be factored into two linear terms whenever the discriminant b^2-4ac is ≥ 0.

When $b^2-4ac<0$ we say that the quadratic ax^2+bx+c is irreducible.

⊛ If ax^2+bx+c factors as $(a_1x+b_1)(a_2x+b_2)$ then $x=-b_1/a_1$ and $x=-b_2/a_2$ are solutions to the equation $ax^2+bx+c=0$. (But note that $-b_1/a_1$ and $-b_2/a_2$ will be 'repeated' roots if $-b_1/a_1 = -b_2/a_2$.)

$f(x) = P(x)/Q(x)$ where $P(x)$, $Q(x)$ are polynomials.

$f(x)$ is <u>proper</u> when $\text{degree}(P) < \text{degree}(Q)$

Integrating Rational Functions (section 7.4)

① There is a family of special rational functions known as "partial fractions".

② Every proper rational function can be expressed as a sum of partial fractions.

③ Previous integration techniques can be used to integrate any partial fraction.

So in practice there are three points to discuss:

① What is a partial fraction?

② Given $P(x)/Q(x)$ with $\text{degree}(P) < \text{degree}(Q)$ how can we write it as a sum of partial fractions?

③ How do you integrate a partial fraction?

Comment: The hardest point is ②!
So we'll discuss that last.

Also Need to circle back to non-proper rational functions.

Additonal Comment:

While the approach outlined in Section 7.4 gives a technique for integrating any rational function, in a specific example there may be a simpler way to calculate the integral.

So, before starting on a long process to work it's a good idea to ask yourself if there might be an easier way to solve the problem.

Example

$$\int \frac{8x^3+3x^2-2}{(2x^4+x^3-2x-1)^2}\,dx = ?$$

We can work this quickly by substituting

$$\begin{cases} u = 2x^4+x^3-2x-1 \\ du = (8x^3+3x^2-2)\,dx \end{cases}$$

to get

$$\int \frac{1}{u^2}\,du = -u^{-1}+C = \frac{1}{2x^4+x^3-2x+1} + C$$

This is much, much easier than discovering the partial fractions decomposition:

$$\frac{8x^3+3x^2-2}{(2x^4+x^3-2x-1)^2} = \frac{1/9}{(x-1)^2} - \frac{16/9}{(2x+1)^2} + \frac{-\frac{1}{3}x-\frac{2}{3}}{(x^2+x+1)^2} + \frac{1/3}{x^2+x+1}$$

A <u>partial fraction</u> is a rational function having one of the forms:

$$\frac{A}{(ax+b)^i} \quad \leftarrow \text{"linear form"}$$

or

$$\frac{Ax+B}{(ax^2+bx+c)^i} \quad \leftarrow \text{"quadratic form"}$$

where A, B, a, b, c, i are constants for which i is a positive integer and ax^2+bx+c is an irreducible quadratic.

<u>Theorem</u> Every proper rational function $f(x)$ can be expressed in one and only one way as a sum of partial fractions.

<u>example</u> $\dfrac{1/3}{x^2+x+1}$ is a partial fraction of quadratic form.

Here $A=0$, $B=1/3$, $a=1$, $b=1$, $c=1$, $i=1$ and discriminant of x^2+x+1 is $-3<0$

$\Rightarrow x^2+x+1$ is irreducible

examples of quadratics :

① $4x^2-8x-5$ has discriminant $(-8)^2-4(4)(-5)=144>0$

$\Rightarrow$ $4x^2-8x-5$ is a reducible quadratic

$\Rightarrow$ it can be factored into linear pieces

In fact, $4x^2-8x-5 = (2x+1)(2x-5)$

② $4x^2-8x+5$ has discriminant $(-8)^2-4(4)(5) = -16<0$

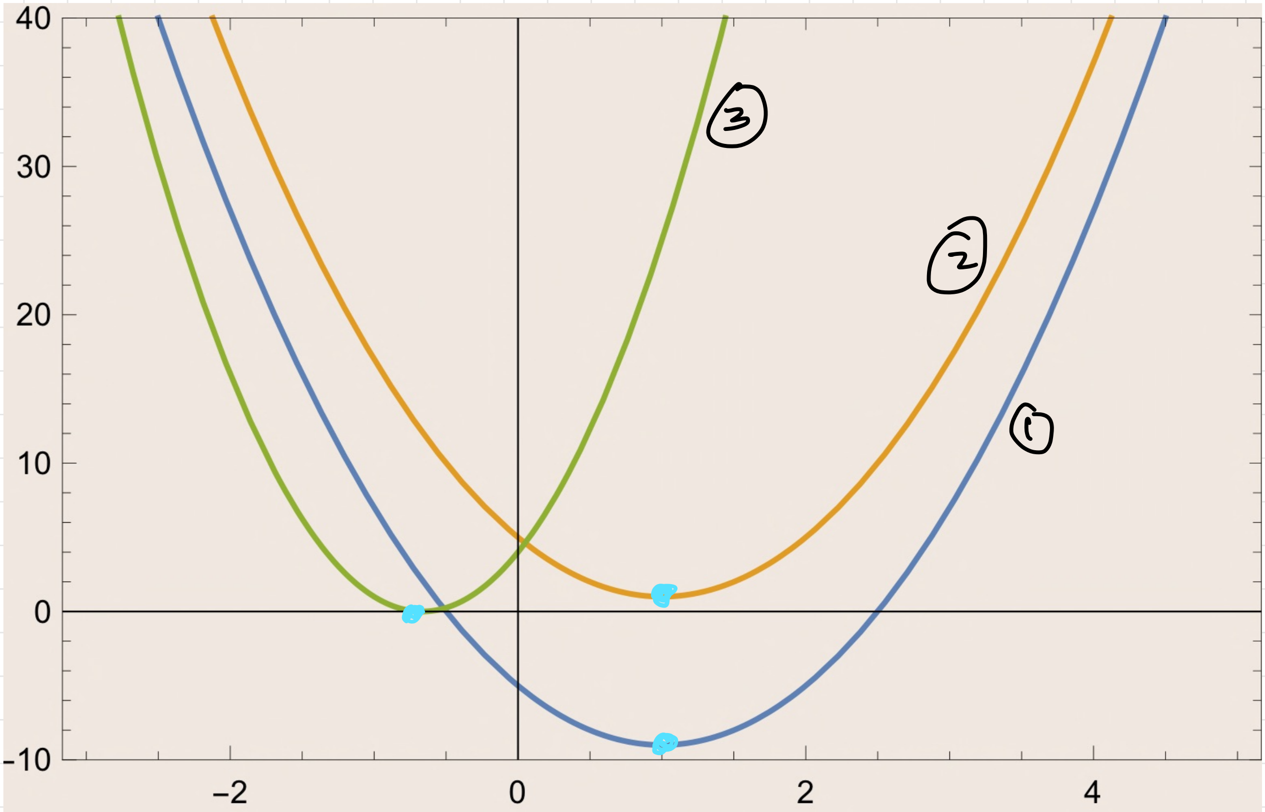
$\Rightarrow$ $4x^2-8x+5$ is an irreducible quadratic.

We can write $4x^2-8x+5 = 4(x^2-2x+1)-4+5 = 4(x-1)^2+1$

③ $9x^2+12x+4$ has discriminant $12^2-4(9)(4)=0$

$\Rightarrow$ $9x^2+12x+4$ is a reducible quadratic

In fact, it is a perfect square $9x^2+12x+4=(3x+2)^2$



For $a>0$ the graph of $y=ax^2+bx+c$ is an upward opening parabola, and its vertex is

$$\begin{cases} \text{below the x-axis when } b^2-4ac>0 & \Rightarrow \text{2 x-intercepts} \\ \text{on the x-axis when } b^2-4ac=0 & \Rightarrow \text{1 x-intercept} \\ \text{above the x-axis when } b^2-4ac<0 & \Rightarrow \text{No x-intercept} \end{cases}$$

For $a<0$, it's a downward opening parabola ....

From the examples ax^2+bx+c on previous page, is $\frac{1}{ax^2+bx+c}$ a partial fraction? and $\int \frac{1}{ax^2+bx+c}\,dx = ?$

① $\frac{1}{4x^2-8x-5}$ No (see next page)

② $\frac{1}{4x^2-8x+5} = \frac{1}{4(x-1)^2+1}$ Yes

$$\int \frac{1}{4(x-1)^2+1}\,dx = \frac{1}{4}\int \frac{1}{(x-1)^2+(1/2)^2}\,dx$$

$$= \frac{1}{4}\int \frac{1}{\frac{1}{4}\cancel{\sec^2\theta}}\,\frac{1}{2}\cancel{\sec^2\theta}\,d\theta$$

← make trig sub

$$\begin{cases} x-1 = \frac{1}{2}\tan\theta \\ dx = \frac{1}{2}\sec^2\theta\,d\theta \end{cases}$$

$$(x-1)^2+\left(\tfrac{1}{2}\right)^2 = \frac{1}{4}\sec^2\theta$$

$$= \frac{1}{2}\theta + C$$

$$= \frac{1}{2}\arctan(2x-1) + C$$

③ $\frac{1}{9x^2+12x+4} = \frac{1}{(3x+2)^2}$ Yes $i=2$, $ax+b = 3x+2$

$$\frac{1}{3}\int \frac{1}{(3x+2)^2}\,3\,dx \leftarrow \begin{cases} u=3x+2 \\ du = 3dx \end{cases}$$

$$= \frac{1}{3}\int \frac{1}{u^2}\,du = -\frac{1}{3}\,\frac{1}{3x+2} + C$$

Conclude: Partial Fractions with linear form (as in ③) are generally easier to integrate than those with quadratic form (as in ②).

Integrating a partial fraction of quadratic form may involve 'completing the square' and/or making a trig substitution $x-b = a\tan\theta$.

(1) $\dfrac{1}{4x^2-8x-5}$ ← This is not a partial fraction but it can be written as a sum of partial fractions. How ?

Step 1 Factor the Denominator:

$$4x^2-8x-5=(2x+1)(2x-5)$$

Step 2 That determines the form of the decomposition:

$$\frac{1}{4x^2-8x-5}=\frac{A}{2x+1}+\frac{B}{2x-5}$$

Step 3 Solve for constants (A and B here)

$$\frac{1+0x}{4x^2-8x-5}=\frac{A(2x-5)+B(2x+1)}{(2x+1)(2x-5)}=\frac{(-5A+B)+(2A+2B)x}{(2x+1)(2x-5)}$$

$$\Rightarrow \begin{cases}-5A+B=1 & (i)\\ 2A+2B=0 & (ii)\end{cases} \Rightarrow B=-A \text{ by (ii)} \Rightarrow -6A=1 \text{ by (i)}$$

So $A=-1/6$, $B=1/6$ and

$$\boxed{\frac{1}{4x^2-8x-5}=\frac{-1/6}{2x+1}+\frac{1/6}{2x-5}}$$

Step 4 Ready to integrate

$$\int\frac{1}{4x^2-8x-5}\,dx=-\frac{1}{6}\int\frac{1}{2x+1}\,dx+\frac{1}{6}\int\frac{1}{2x-5}\,dx$$

$$=-\frac{1}{12}\ln|2x+1|+\frac{1}{12}\ln|2x-5|+C$$

$$=\frac{1}{12}\ln\left|\frac{2x-5}{2x+1}\right|+C$$ (if you want)

Fact Every irreducible quadratic can be written in the form $C\left((x+b)^2 + a^2\right)$.

example

$\frac{3-2x}{(x+1)^2}$ is not a partial fraction but

$$\frac{3-2x}{(x+1)^2} = \frac{-2}{x+1} + \frac{5}{(x+1)^2}$$ where

$\frac{-2}{x+1}$ and $\frac{5}{(x+1)^2}$ are partial fractions.

example

$\frac{3-2x}{(x^2+1)^2}$ is a partial fraction.