Which are partial fractions?

① $\dfrac{7}{\sqrt{2x+1}}$ N $\qquad \dfrac{7}{(2x+1)^{1/2}}$

② $\dfrac{7}{(2x+1)^2}$ Y $\qquad$ linear form

③ $\dfrac{7}{(2x^2+1)^3}$ Y $\qquad$ quadratic form

$2x^2+1 = 2x^2+0x+1 \Rightarrow b^2-4ac = 0^2-4(2)(1)$
$= -8 < 0$

④ $\dfrac{7}{(2x^2-1)^3}$ N

$0^2-4(2)(-1)=8$
$\Rightarrow 2x^2-1$ is reducible

⑤ $\dfrac{7x-3}{(2x^2-1)^3}$ N

⑥ $\dfrac{7x-3}{(2x-1)^3}$ N

⑦ $\dfrac{x}{(3x+1)^4}$ N

⑧ $\dfrac{x}{(x^2+x+1)^4}$ Y $\qquad$ quadratic form

$b^2-4ac = 1-4(1)(1)=-3$
$\Rightarrow x^2+x+1$ is irreducible

⑨ $\dfrac{1}{3x^3+x+1}$ N

Partial Fractions:

$$\frac{A}{(ax+b)^i} \quad \text{or} \quad \frac{Ax+B}{(ax^2+bx+c)^i}$$

'linear form' 'quadratic form'

where A, B, a, b, c, i are constants for which i is a positive integer and ax^2+bx+c is an irreducible quadratic.

Theorem Every proper rational function $f(x)$ can be expressed in one and only one way as a sum of partial fractions.

How to write $\frac{P(x)}{Q(x)}$ as a sum of partial fractions.

Step 1 Factor the denominator.

(algebra problem, can be nigh impossible)

Step 2 Determine the form of the sum.

(immediate)

Step 3 Solve for constants.

(straightforward but may be tedious)

Step 4 Ready to calculate the integral.

(linear terms are easier than quadratic)

Goal: Calculate $\int \frac{P(x)}{Q(x)}\,dx$

Examples for Step 2: Determine the form of the sum. (where denominator is already factored into linear and irreducible quadratic terms).

① $$\frac{x^2+1}{(x+1)(x+2)(x+3)} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x+3}$$

② $$\frac{x^2+1}{(x+1)^3(x+2)} = \left(\frac{A}{(x+1)^3} + \frac{B}{(x+1)^2} + \frac{C}{x+1}\right) + \frac{D}{x+2}$$

multiplicity $=3 \Rightarrow$ 3 terms here

③ $$\frac{x^2+3}{(x^2+1)\,x} = \frac{Ax+B}{x^2+1} + \frac{C}{x}$$

④ $$\frac{x^2+3}{(x^2+x+1)^2\,x^2} = \left(\frac{Ax+B}{(x^2+x+1)^2} + \frac{Cx+D}{x^2+x+1}\right) + \left(\frac{E}{x^2} + \frac{F}{x}\right)$$

mult $=2$ mult $=2$ 2 terms 2 terms

challenge: In ④,

$A=3, B=1, C=6, D=3, E=3, F=-6$

Illustration for carrying out

Step 3: Solve for constants.

(3) $$\frac{x^2+3}{(x^2+1)x} = \frac{Ax+B}{x^2+1} + \frac{C}{x}$$

Goal: Determine A, B, C

$$\frac{x}{x}\cdot\frac{Ax+B}{x^2+1} + \frac{C}{x}\cdot\frac{x^2+1}{x^2+1} = \frac{(Ax+B)x + C(x^2+1)}{(x^2+1)x}$$

$$= \frac{(A+C)x^2 + Bx + C}{(x^2+1)x} = \frac{x^2+0x+3}{(x^2+1)x}$$

$$\Rightarrow \begin{cases} A+C=1 \\ B=0 \\ C=3 \end{cases} \Rightarrow A = 1-C = -2$$

$A=-2$
$B=0$
$C=3$

so $$\boxed{\frac{x^2+3}{(x^2+1)x} = \frac{-2x}{x^2+1} + \frac{3}{x}}$$

$$\int \frac{x^2+3}{(x^2+1)x}\,dx = \int \frac{-2x}{x^2+1} + \frac{3}{x}\,dx = -\ln(x^2+1) + 3\ln|x| + C$$

example

$$\int \frac{1}{(x+1)^2(x-1)}\,dx = ?$$

mult 2   mult 1

Step 1 Factor the denominator.
Step 2 Determine the form of the sum.
Step 3 Solve for constants.
Step 4 Ready to calculate the integral.

Step 1 ✓ (already factored)

Step 2 : $\frac{1}{(x+1)^2(x-1)} = \frac{A}{(x+1)^2}\frac{x-1}{x-1} + \frac{B}{x+1} + \frac{C}{x-1}$

Step 3 :

$$\frac{0x^2+0x+1}{(x+1)^2(x-1)} = \frac{A(x-1)+B(x+1)(x-1)+C(x+1)^2}{(x+1)^2(x-1)}$$

$$= \frac{(\overset{0}{B+C})x^2+(\overset{0}{A+2C})x+(\overset{1}{-A-B+C})}{(x+1)^2(x-1)}$$

$$\Rightarrow \begin{cases} B+C=0 \\ A+2C=0 \\ -A-B+C=1 \end{cases} \Rightarrow \begin{cases} B=-C \\ A=-2C \\ 4C=1 \end{cases} \Rightarrow \begin{cases} C=1/4 \\ B=-1/4 \\ A=-1/2 \end{cases}$$

$-(-2C)-(-C)+C=1$

Step 4 :

$$\int \frac{\overset{A}{-1/2}}{(x+1)^2} + \frac{\overset{B}{-1/4}}{x+1} + \frac{\overset{C}{1/4}}{x-1}\,dx$$

$$= \frac{1}{2}\frac{1}{x+1} - \frac{1}{4}\ln|x+1| + \frac{1}{4}\ln|x-1| + C$$

Example (continued)

Since $(x+1)^2(x-1) = x^3+x^2-x-1$ the previous integral might have been given as

$$\int \frac{1}{x^3+x^2-x-1}\,dx$$

in which case we would have needed to start with:

Step 1 Factor x^3+x^2-x-1.

How might we have done that? Factoring degree 3 polynomials is not easy but we could observe that $x=1 \Rightarrow x^3+x^2-x-1=0$ which means that $(x-1)$ is a factor of x^3+x^2-x-1. Either

① We could write

$$x^3+x^2-x-1 = (x-1)(ax^2+bx+c) = ax^3+(b-a)x^2+(c-b)x-c$$

to get $a=c=1, b=2$, or

② Do long division

$$\begin{array}{r|l}
 & x^2+2x+1 \\
\hline
x-1 & x^3+x^2-x-1 \\
 & \underline{x^3-x^2} \\
 & \quad 2x^2-x \\
 & \quad \underline{2x^2-2x} \\
 & \qquad x-1 \\
 & \qquad \underline{x-1} \\
 & \qquad\quad 0
\end{array}$$

Conclusion:

$$x^3+x^2-x+1 = (x-1)(x^2+2x+1) = (x-1)(x+1)^2$$

7.5 EXERCISES

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1–82 Evaluate the integral.

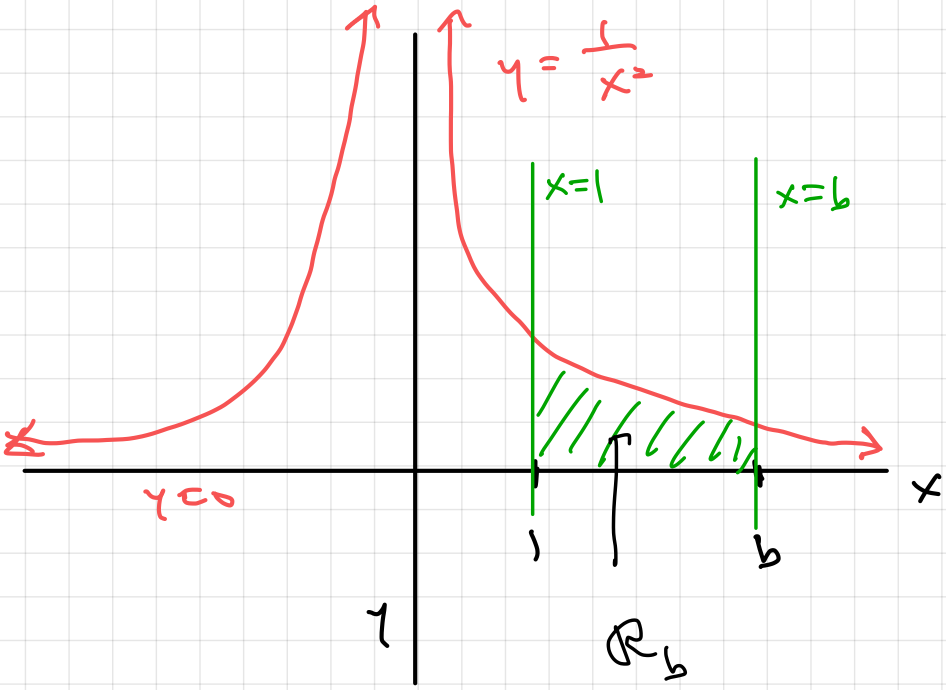
1. $\int \frac{\cos x}{1-\sin x}\,dx$
2. $\int_0^1 (3x+1)^{\sqrt{2}}\,dx$
3. $\int_1^4 \sqrt{y}\ln y\,dy$
4. $\int \frac{\sin^3 x}{\cos x}\,dx$
5. $\int \frac{t}{t^4+2}\,dt$
6. $\int_0^1 \frac{x}{(2x+1)^3}\,dx$
7. $\int_{-1}^1 \frac{e^{\arctan y}}{1+y^2}\,dy$
8. $\int t\sin t\cos t\,dt$
9. $\int_2^4 \frac{x+2}{x^2+3x-4}\,dx$
10. $\int \frac{\cos(1/x)}{x^3}\,dx$
11. $\int \frac{1}{x^3\sqrt{x^2-1}}\,dx$
12. $\int \frac{2x-3}{x^3+3x}\,dx$
13. $\int \sin^5 t\cos^4 t\,dt$
14. $\int \ln(1+x^2)\,dx$
15. $\int x\sec x\tan x\,dx$
16. $\int_0^{\sqrt{2}/2} \frac{x^2}{\sqrt{1-x^2}}\,dx$
17. $\int_0^\pi t\cos^2 t\,dt$
18. $\int_1^4 \frac{e^{\sqrt{t}}}{\sqrt{t}}\,dt$
19. $\int e^{x+e^x}\,dx$
20. $\int e^2\,dx$
21. $\int \arctan\sqrt{x}\,dx$
22. $\int \frac{\ln x}{x\sqrt{1+(\ln x)^2}}\,dx$
23. $\int_0^1 (1+\sqrt{x})^8\,dx$
24. $\int (1+\tan x)^2\sec x\,dx$
25. $\int_0^1 \frac{1+12t}{1+3t}\,dt$
26. $\int_0^1 \frac{3x^2+1}{x^3+x^2+x+1}\,dx$
27. $\int \frac{dx}{1+e^x}$
28. $\int \sin\sqrt{at}\,dt$
29. $\int \ln\left(x+\sqrt{x^2-1}\right)dx$
30. $\int_{-1}^2 |e^x-1|\,dx$
31. $\int \sqrt{\frac{1+x}{1-x}}\,dx$
32. $\int_1^3 \frac{e^{3/x}}{x^2}\,dx$
33. $\int \sqrt{3-2x-x^2}\,dx$
34. $\int_{\pi/4}^{\pi/2} \frac{1+4\cot x}{4-\cot x}\,dx$
35. $\int_{-\pi/2}^{\pi/2} \frac{x}{1+\cos^2 x}\,dx$
36. $\int \frac{1+\sin x}{1+\cos x}\,dx$
37. $\int_0^{\pi/4} \tan^3\theta\sec^2\theta\,d\theta$
38. $\int_{\pi/6}^{\pi/3} \frac{\sin\theta\cot\theta}{\sec\theta}\,d\theta$
39. $\int \frac{\sec\theta\tan\theta}{\sec^2\theta-\sec\theta}\,d\theta$
40. $\int_0^\pi \sin 6x\cos 3x\,dx$
41. $\int \theta\tan^2\theta\,d\theta$
42. $\int \frac{\tan^{-1}x}{x^2}\,dx$
43. $\int \frac{\sqrt{x}}{1+x^3}\,dx$
44. $\int \sqrt{1+e^x}\,dx$
45. $\int x^5 e^{-x^3}\,dx$
46. $\int \frac{(x-1)e^x}{x^2}\,dx$
47. $\int x^3(x-1)^{-4}\,dx$
48. $\int_0^1 x\sqrt{2-\sqrt{1-x^2}}\,dx$

Color Key:

- = See 4/14 class notes
- = rational functions (only #25 is not proper)
- = trig substitution
- = rework to get a rational function.

$$\frac{x^3}{(x-1)^4} = \frac{A}{(x-1)^4} + \frac{B}{(x-1)^3} + \frac{C}{(x-1)^2} + \frac{D}{x-1}$$

but substituting $u = x-1$ maybe easier?



$b > 1$

For $b > 1$, find the area of the region R_b under $y = 1/x^2$ in first quadrant where $1 \le x \le b$.

$$\text{area}(R_b) = \int_1^b \frac{1}{x^2} - 0 \, dx = -\frac{1}{x}\Big|_1^b = -\frac{1}{b} + 1$$

Notice that $\text{area}(R_b) < 1$ and that $\text{area}(R_b)$ limits to 1 as b goes to ∞. Conclude: The infinitely extending region

$$R: \begin{cases} 0 \le y \le 1/x^2 \\ 1 \le x < \infty \end{cases}$$

has $\text{area}(R) = 1$. We will write

$$\int_1^\infty \frac{1}{x^2} \, dx = \lim_{b \to \infty} \int_1^b \frac{1}{x^2} \, dx = 1$$

This is called an <u>improper integral</u>.

<u>Definition</u>

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$$

and if this limit equals $L \neq \pm\infty$ then we say that the improper integral $\int_a^\infty f(x)\,dx$ <u>converges</u> to L.

<u>example</u> $\int_1^\infty \frac{1}{x}\,dx = \infty$

So this integral doesn't converge.

<u>example</u> $\int_1^\infty \frac{1}{e^x}\,dx = ?$

$$\int_1^b \frac{1}{e^x}\,dx = \int_1^b e^{-x}\,dx = -e^{-x}\Big|_1^b = -e^{-b} + e^{-1}$$

$$\lim_{b\to\infty} \left(-e^{-b} + e^{-1}\right) = \lim_{b\to\infty} -\frac{1}{e^b} + \frac{1}{e} = \frac{1}{e}$$

$$\Rightarrow \int_1^\infty \frac{1}{e^x}\,dx = \frac{1}{e} \qquad \text{(converges)}$$