# Problems : Find the limits

$f(x) = 7x - 14$ , $g(x) = 3x - 6$ , $h(x) = 7x + 5$

- $\lim_{x \to \infty} f(x) =$
- $\lim_{x \to \infty} h(x) =$
- $\lim_{x \to \infty} f(x) + h(x) =$
- $\lim_{x \to \infty} f(x) - h(x) =$
- $\lim_{x \to \infty} \frac{f(x)}{h(x)} =$
- $\lim_{x \to 2} g(x) =$
- $\lim_{x \to 2} \frac{1}{f(x)} =$
- $\lim_{x \to 2+} \frac{1}{f(x)} =$
- $\lim_{x \to 2+} \frac{1}{f(x)} g(x) =$
- $\lim_{x \to 2} \frac{g(x)}{f(x)} =$

Each of these limits can be calculated using basic algebra.

(answers to be given below)

**Problem** $f(x) = \frac{1}{1+x^2}$

length $1-(-1) = 2$

ⓐ Find the average value of $f(x)$ on $[-1,1]$.

length $2a$

ⓑ Find the limit of the average value of $f$ on $[-a,a]$ as $a \to +\infty$.

ⓐ $f_{ave} = \frac{1}{2}\int_{-1}^{1} \frac{1}{1+x^2}\,dx = \frac{1}{2}\arctan(x)\Big|_{x=-1}^{1}$

$= \frac{1}{2}\left(\arctan(1) - \arctan(-1)\right) = \frac{1}{2}\left(\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right) = \pi/4$

ⓑ $f_{ave} = \frac{1}{2a}\int_{-a}^{a} \frac{1}{1+x^2}\,dx = \frac{1}{2}\arctan(x)\Big|_{x=-a}^{a}$

$= \frac{1}{2a}\left(\arctan(a) - \arctan(-a)\right) = \frac{1}{a}\arctan(a)$

$\lim_{a\to\infty} \frac{1}{a}\arctan(a) = \lim_{a\to\infty} \overbrace{\frac{1}{a}}^{0} \lim_{a\to\infty} \overbrace{\arctan(x)}^{\pi/2} =$

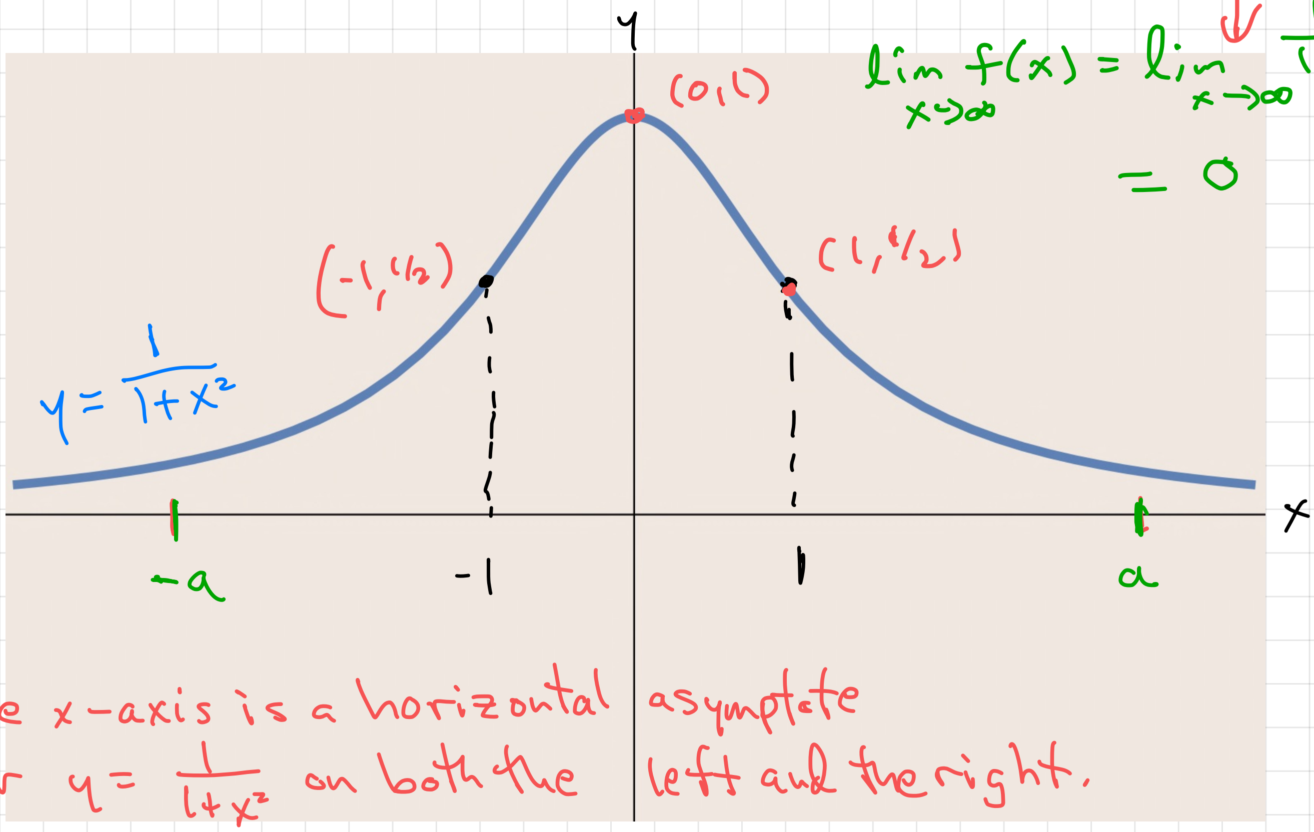
$= (0)\cdot(\pi/2) = 0$

form: $1/\infty = 0$

Does this answer make sense? Look at graph

$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{1}{1+x^2}$

$= 0$




The x-axis is a horizontal asymptote for $y = \frac{1}{1+x^2}$ on both the left and the right.

For limits involving $\pm\infty$, $0$ or numbers $L, M \neq 0$ forms can be either determinate or indeterminate:

Some determinate forms are:

$\infty + \infty$, $\infty \cdot \infty$, $\frac{L}{\infty}$, $\frac{0}{\infty}$, $0+0$, $0+\infty$, $\infty(-\infty) = -\infty$, $(-\infty)\cdot(-\infty) = \infty$, etc

Some indeterminate forms are:

$\frac{\infty}{\infty}$, $\frac{0}{0}$, $\infty - \infty$, $0 \cdot \infty$, $1^\infty$, $\infty^0$, $0^0$

If we encounter an indeterminate form when calculating the limit of an expression what options are there?

- Use algebra to rewrite the expression
- Use L'Hospital's Rule
- Use a combination of the above

intuition Think of

$\infty$ = a really really large positive number

$-\infty$ = a really really large negative number

$0$ = a number very close to 0

but realize $\infty$ and $-\infty$ are relative concepts,

DANGER It's easy to think that $+\infty$ and $-\infty$ are numbers and satisfy all laws of arithmetic — they don't !!

example:

$0 \cdot \infty$ = indeterminate form

because the product of a number close to 0 with a large number could be either large or small.

example $10^{-12}$ ← close to 0, $10^{9}$ ← very large

$(10^{-12})(10^{9}) = 10^{-3}$ small

example $10^{-9}$ ← close to 0, $10^{12}$ ← very large

$(10^{-9})(10^{12}) = 10^{3}$ large

Problems : Find the limits using algebra.

$f(x) = 7x - 14$ , $g(x) = 3x - 6$ , $h(x) = 7x + 5$

- $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} 7x - 14 = +\infty$
- $\lim_{x\to\infty} h(x) = +\infty$

form $\infty + \infty = \infty$

- $\lim_{x\to\infty} f(x) + h(x) = \lim_{x\to\infty} 14x - 9 = +\infty$

form $\infty - \infty$

- $\lim_{x\to\infty} f(x) - h(x) = \lim_{x\to\infty} (7x-14) - (7x-5) = -9$

form $\infty/\infty$

- $\lim_{x\to\infty} \frac{f(x)}{h(x)} = \lim_{x\to\infty} \frac{7x-14}{7x+5} = \lim_{x\to\infty} \frac{7 - 14/x}{7 + 5/x} = 1$
- $\lim_{x\to 2} g(x) = g(2) = 0$
- $\lim_{x\to 2} \frac{1}{f(x)} = \text{DNE}$
- $\lim_{x\to 2+} \frac{1}{f(x)} = +\infty = \frac{1}{0^+}$
- $\lim_{x\to 2+} \frac{1}{f(x)} g(x) = \lim_{x\to 2+} \frac{3x-6}{7x-14} = 3/7$

form $0 \cdot \infty$

- $\lim_{x\to 2} \frac{g(x)}{f(x)} = \lim_{x\to 2} \frac{3x-6}{7x-14} = \lim_{x\to 2} \frac{3\cancel{(x-2)}}{7\cancel{(x-2)}} = 3/7$

Recall: In Calculus I you would have learned various facts about limits, including:

$$\boxed{\text{If } \lim_{x \to a} f(x) = L \text{ and } \lim_{x \to a} g(x) = M \text{ then } \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{L}{M} \text{ as long as } M \neq 0.}$$

But what happens if $M = 0$.?

It can be observed that

$$\boxed{\text{If } M = 0 \text{ and } L \neq 0 \text{ then } \lim_{x \to a} \frac{f(x)}{g(x)} = \text{DNE}.}$$

Although in some cases DNE be replaced by $\infty$ or $-\infty$.

In fact we consider

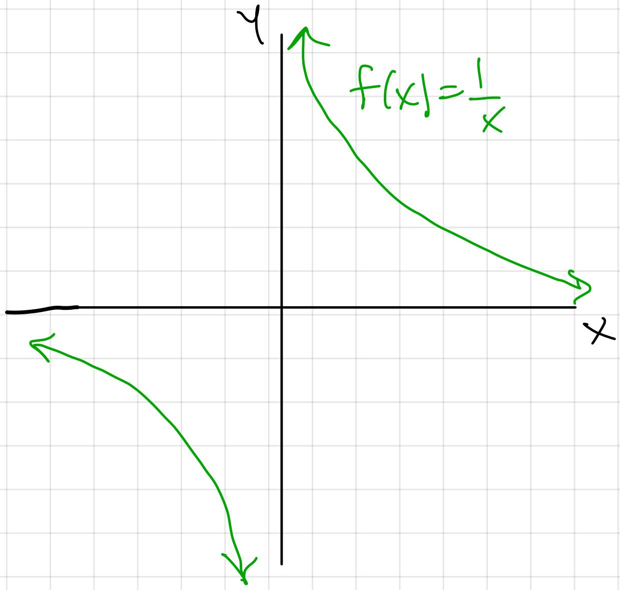
$$\boxed{\text{If } L \text{ is a positive number then } \frac{L}{0+} = \infty \text{ and } \frac{L}{0^+} = -\infty.}$$

to be determinate forms. Think of 0+ as representing a positive number very close to 0 and 0− as a negative number very close to 0.

examples →

example

$$\lim_{x\to 0} \frac{1}{x} = \text{DNE}$$

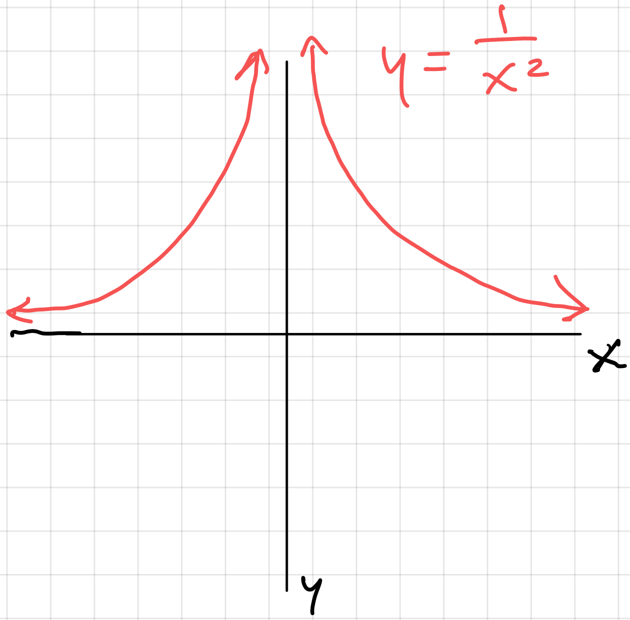



$$\lim_{x\to 0+} \frac{1}{x} = +\infty = \frac{1}{0+}$$

$$\lim_{x\to 0-} \frac{1}{x} = -\infty = \frac{1}{0-}$$

$\Rightarrow$ The two-sided limit $\lim_{x\to 0} \frac{1}{x}$ doesn't exist because the left and right hand limits $\lim_{x\to 0+} 1/x$ and $\lim_{x\to 0-}$ are not equal.

example




$$\lim_{x\to 0} \frac{1}{x^2} = \infty$$

This has $\frac{1}{0+}$ form.

**L'Hospital's Rule** Suppose $f$ and $g$ are differentiable and $g'(x) \neq 0$ on an open interval $I$ that contains $a$ (except possibly at $a$). Suppose that

$$\lim_{x \to a} f(x) = 0 \quad \text{and} \quad \lim_{x \to a} g(x) = 0$$

or that

$$\lim_{x \to a} f(x) = \pm\infty \quad \text{and} \quad \lim_{x \to a} g(x) = \pm\infty$$

(In other words, we have an indeterminate form of type $\frac{0}{0}$ or $\infty/\infty$.) Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is $\infty$ or $-\infty$).

example Let $f(x) = \ln(1+x)$, $g(x) = x$

$f'(x) = \frac{1}{1+x}$
$g'(x) = 1$

then $\lim_{x\to 0} f(x) = 0$ and $\lim_{x \to 0} g(x) = 0$

$\ln(1+0)$

So $\displaystyle\lim_{x\to 0} \frac{\ln(1+x)}{x} \overset{L'H}{=} \lim_{x\to 0} \frac{\frac{1}{1+x}}{1} = \lim_{x\to 0} \frac{1}{1+x} = 1$

$\frac{0}{0}$ form

example $\displaystyle\lim_{x\to\infty} \frac{e^x}{x^2} = \infty$

$\infty/\infty$ form
use L'Hospital twice here.

$f(x) = e^x$, $f'(x) = e^x$
$g(x) = x^2$, $g'(x) = 2x$

$$\lim_{x\to\infty} \frac{e^x}{x^2} \overset{L'H}{=} \lim_{x\to\infty} \frac{e^x}{2x} = \frac{1}{2} \lim_{x\to\infty} \frac{e^x}{x}$$

$$\overset{L'H}{=} \frac{1}{2} \lim_{x\to\infty} \frac{e^x}{1} = \infty$$