For limits involving $\pm\infty$, 0 or numbers $L, M \neq 0$ forms can be either determinate or indeterminate.

Some determinate forms are:

use your intuition to remember these

$\infty + \infty = \infty$

$\infty \cdot \infty = \infty$

$L/\infty = 0$

$0 + \infty = \infty$

$1/0_+ = \infty$

$\infty \cdot (-\infty) = -\infty$

$(-\infty)\cdot(-\infty) = \infty$

$0 + 0 = 0$

$0^\infty = 0$

$1/0_- = -\infty$

$$L \cdot \infty = \begin{cases} \infty & \text{if } L > 0 \\ -\infty & \text{if } L < 0 \end{cases}$$

$$L^\infty = \begin{cases} \infty & \text{if } L > 1 \\ 0 & \text{if } 0 < L < 1 \end{cases}, \quad \text{etc}$$

Some indeterminate forms are:

$\frac{\infty}{\infty}$, $\frac{0}{0}$, $\infty - \infty$, $0 \cdot \infty$, 1^∞, ∞^0, 0^0

NEITT Be careful with these. ∞ is not really a number!

If we encounter an indeterminate form when calculating the limit of an expression what options are there?

- Use algebra to rewrite the expression
- Use L'Hospital's Rule
- Use a combination of the above

To be clear, each determinate form is a rule about limits. For example,

$0 + \infty = \infty$ means:

$$\boxed{\text{If } \lim_{x \to a} f(x) = 0 \text{ and } \lim_{x \to a} g(x) = \infty \text{ then } \lim_{x \to a} f(x) + g(x) = \infty.}$$

$$\boxed{\text{If } \lim_{x \to a} f(x) = 1 \text{ and } \lim_{x \to a} g(x) = 0 \text{ where } g(x) > 0 \text{ for all values of } x \text{ close to } a \text{ but not equal to } a \text{ then } \lim_{x \to a} \frac{f(x)}{g(x)} = \infty}$$

L'Hospital's Rule Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a). Suppose that

$$\lim_{x\to a} f(x) = 0 \quad \text{and} \quad \lim_{x\to a} g(x) = 0 \qquad 0/0$$

or that

$$\lim_{x\to a} f(x) = \pm\infty \quad \text{and} \quad \lim_{x\to a} g(x) = \pm\infty \qquad \pm\infty/\pm\infty$$

(In other words, we have an indeterminate form of type $\frac{0}{0}$ or ∞/∞.) Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is ∞ or $-\infty$).

example $\displaystyle\lim_{x\to\infty} \frac{x^3}{e^x}$ has form ∞/∞, so L'Hospital's Rule applies:

∞/∞ form — use L'H again

$$\lim_{x\to\infty} \frac{x^3}{e^x} \overset{L'H}{=} \lim_{x\to\infty} \frac{3x^2}{e^x}$$

$\frac{\infty}{\infty}$ form $\qquad \frac{6}{\infty}$ is determinate!

$$\overset{L'H}{=} \lim_{x\to\infty} \frac{6x}{e^x} \overset{L'H}{=} \lim_{x\to\infty} \frac{6}{e^x} = 0$$

Note This example can be generalized to show that $\displaystyle\lim_{x\to\infty} \frac{x^n}{e^x} = 0$ for any positive integer n.

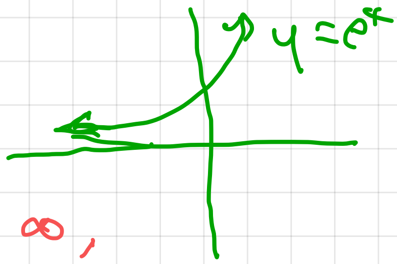
So e^x grows much faster than x^n — this explains "exponential growth".

example

$$\lim_{x\to 0} \frac{\arctan(x)}{x} \overset{L'H}{=} \lim_{x\to 0} \frac{\frac{1}{1+x^2}}{1} = 1$$

0/0 form

example $\lim_{x\to-\infty} x^2 e^x = 0$



This limit has the indeterminate form $0\cdot\infty$, but if we rewrite it algebraically as

$\lim_{x\to-\infty} \frac{x^2}{e^{-x}}$ then it has form ∞/∞ and L'H can be used: $\quad e^x = \frac{1}{e^{-x}}$

$$\lim_{x\to-\infty} \frac{x^2}{e^{-x}} \overset{L'H}{=} \lim_{x\to-\infty} \frac{2x}{-e^{-x}} \qquad \frac{-\infty}{-\infty} \text{ form}$$

$$\overset{L'H}{=} \lim_{x\to-\infty} \frac{2}{e^{-x}} = 0 \qquad \frac{2}{\infty} \text{ form}$$

note: We might alternately have written $x^2 e^x = \frac{e^x}{1/x^2}$ but then you would find that L'Hospital's Rule doesn't produce an answer.

> In this way indeterminate forms $0\cdot\infty$ can be rewritten algebraically to get a form $0/0$ or ∞/∞ where L'H can be applied.
>
> To deal with indeterminate forms 1^∞, ∞^0 or 0^0 we can convert to using L'H by first taking logarithms.

example $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$ ← form 1^∞

Write $y = \left(1+\frac{1}{x}\right)^x$ and take logarithm

$$\ln(y) = \ln\left(1+\frac{1}{x}\right)^x = x\ln\left(1+\frac{1}{x}\right)$$

Now

$$\lim_{x\to\infty}\ln(y) = \lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right) \quad \leftarrow \text{form } 0\cdot\infty$$

$$= \lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{1/x} \quad \leftarrow \text{form } \frac{0}{0}$$

$$\overset{L'H}{=} \lim_{x\to\infty}\frac{\frac{1}{1+\frac{1}{x}}\cdot\cancel{\left(-\frac{1}{x^2}\right)}}{\cancel{\left(-1/x^2\right)}}$$

$$= \lim_{x\to\infty}\frac{1}{1+\frac{1}{x}} = 1$$

So

$$\lim_{x\to\infty} y = \lim_{x\to\infty} e^{\ln(y)} = e^{\lim_{x\to\infty}\ln(y)} = e^1$$

example $\lim_{x\to\infty}\left(1+\frac{3}{x}\right)^x = e^3$ ← 1^∞ form

See if you can work this out using a process similar to the previous example.

Some more examples (Stewart section 6.8)

35. $\displaystyle\lim_{x\to 0} \frac{\ln(1+x)}{\cos x + e^x - 1} = 0$

← for $\frac{0}{1}$

form $0/0$

50. $\displaystyle\lim_{x\to(\pi/2)^-} \overset{0}{\cos x}\ \overset{\infty}{\sec 5x} = \lim_{x\to\pi/2^-} \frac{\cos x}{\cos(5x)}$

$$\overset{L'H}{=} \lim_{x\to\pi/2^-} \frac{-\sin x}{-5\sin(5x)} = 1/5$$

← $\frac{-1}{-5}$ form

$$\sec(5x) = \frac{1}{\cos(5x)}$$

form $\frac{15}{80}$

- $\displaystyle\lim_{x\to 3} \frac{x^3-5x+3}{x^4-1} = \frac{3}{16}$

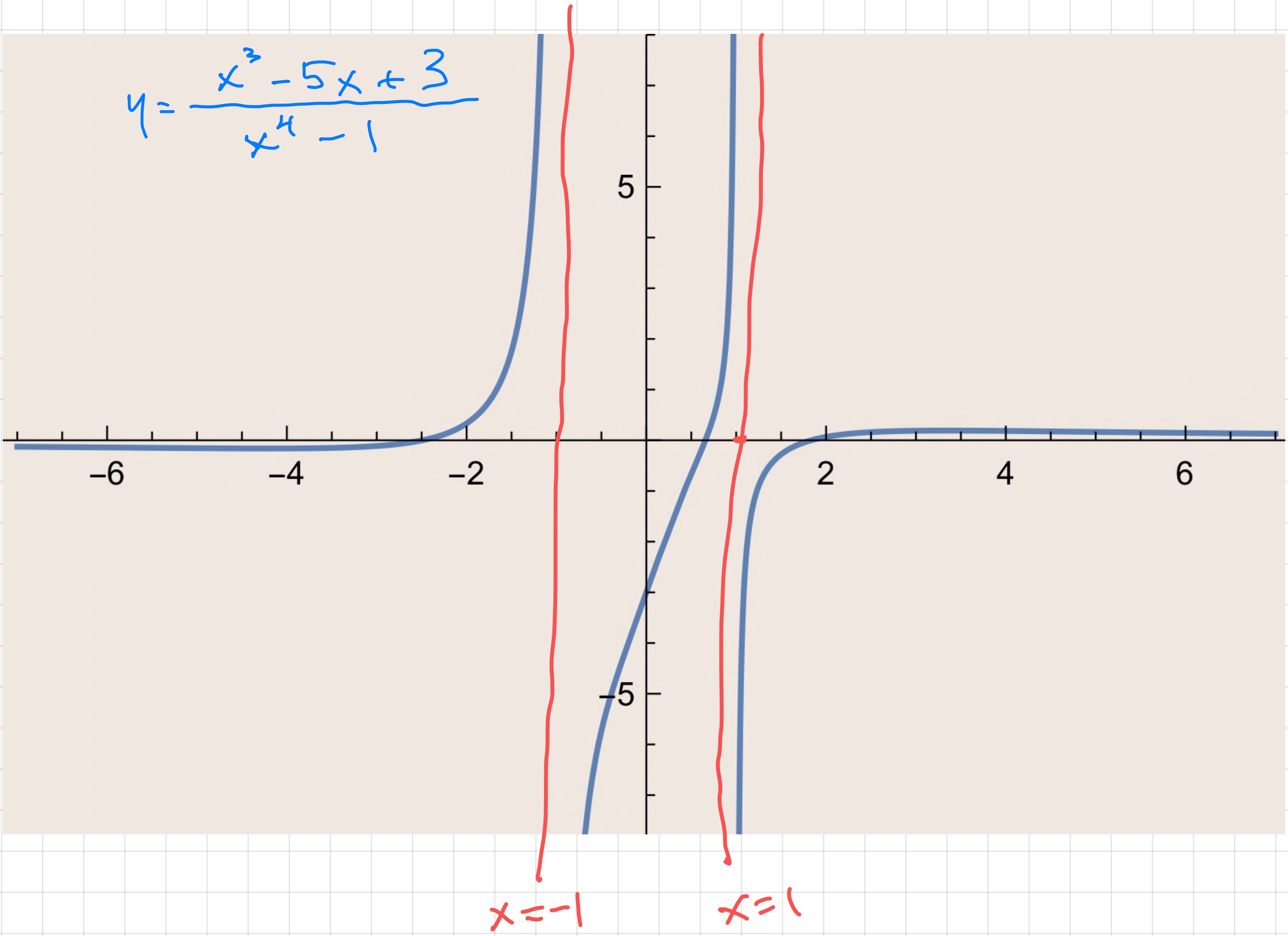
∞/∞ form

- $\displaystyle\lim_{x\to\infty} \frac{x^3-5x+3}{x^4-1} \cdot \frac{1/x^4}{1/x^4} = \lim_{x\to 3} \frac{\overset{0}{\cancel{\frac{1}{x}}} - \overset{0}{\cancel{\frac{5}{x^3}}} + \overset{0}{\cancel{\frac{3}{x^4}}}}{1 - \cancel{1/x^4} \to 0}$

$$= \frac{0}{1} = 0$$

form $\frac{-1}{0}$

- $\displaystyle\lim_{x\to 1} \frac{x^3-5x+3}{x^4-1} = \text{DNE}$



- $\lim_{x \to 3} \frac{x^3 - 5x + 3}{x^4 - 1} = \frac{3}{16}$
- $\lim_{x \to \infty} \frac{x^3 - 5x + 3}{x^4 - 1} = 0$
- $\lim_{x \to 1} \frac{x^3 - 5x + 3}{x^4 - 1} = \text{DNE}$

← Understand these limits in terms of the graph of the rational function.

$$y = \frac{x^3 - 5x + 3}{x^4 - 1}$$

$$\lim_{x \to 1^+} \frac{x^3 - 5x + 3}{x^4 - 1} = -\infty$$

$$\lim_{x \to 1^-} \frac{x^3 - 5x + 3}{x^4 - 1} = \infty$$

← Can you see this in the graph?

Chapter 7 Techniques of Integration.

First Comment: Every rule for differentiation can be written also as a rule for integration but the integration rule is likely to be much more complicated. (Remember: Integration is hard.)

Example
$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

$$\Rightarrow \int \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}\,dx = \frac{f(x)}{g(x)} + C$$

but can we ever recognize when an integral has this form? (not too likely!)

However there is a procedure for turning the product rule into an integration technique — called integration by parts.

Product Rule

$$\frac{d}{dx}[uv] = v\frac{du}{dx} + u\frac{dv}{dx}$$

$$\Rightarrow \quad uv + C = \int v\frac{du}{dx}\,dx + \int u\frac{dv}{dx}\,dx$$
$$= \int v\,du + \int u\,dv$$

And solving for $\int u\,dv$ gives

$$\boxed{\int u\,dv = uv - \int v\,du} \quad \text{IP}$$

comment In this formula we can leave out $+C$ because both integrals in IP have an implied $+C$. .

IP is the formula for "integration by parts".

To use IP one starts with making a choice for u and dv. Then calculate $du = u'\,dx$ and $v = \int dv$.