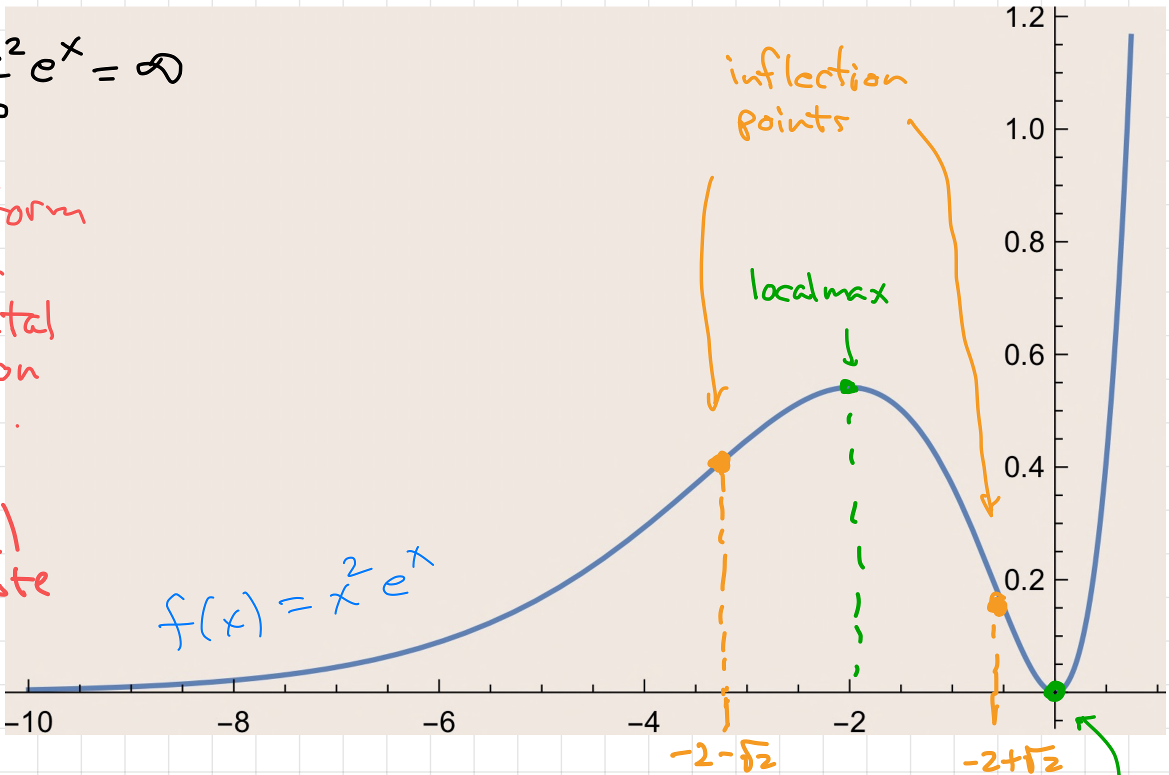
Example We used L'Hospital's Rule to determine that $\lim_{x\to-\infty} x^2e^x = 0$. What does this say about the graph of $f(x)=x^2e^x$?

limit has $0\cdot\infty$ form

$\lim_{x\to\infty} x^2e^x = \infty$

limit has form $\infty\cdot\infty=\infty$. No horizontal asymptote on the right.

$y=0$ is horizontal asymptote on left.



absolute min

$f'(x) = 2xe^x + x^2e^x = xe^x(2+x)$

critical numbers for $f(x)$: $x=0$, $x=-2$

f inc		f dec		f inc
$f'>0$	-2	$f'<0$	0	$f'>0$

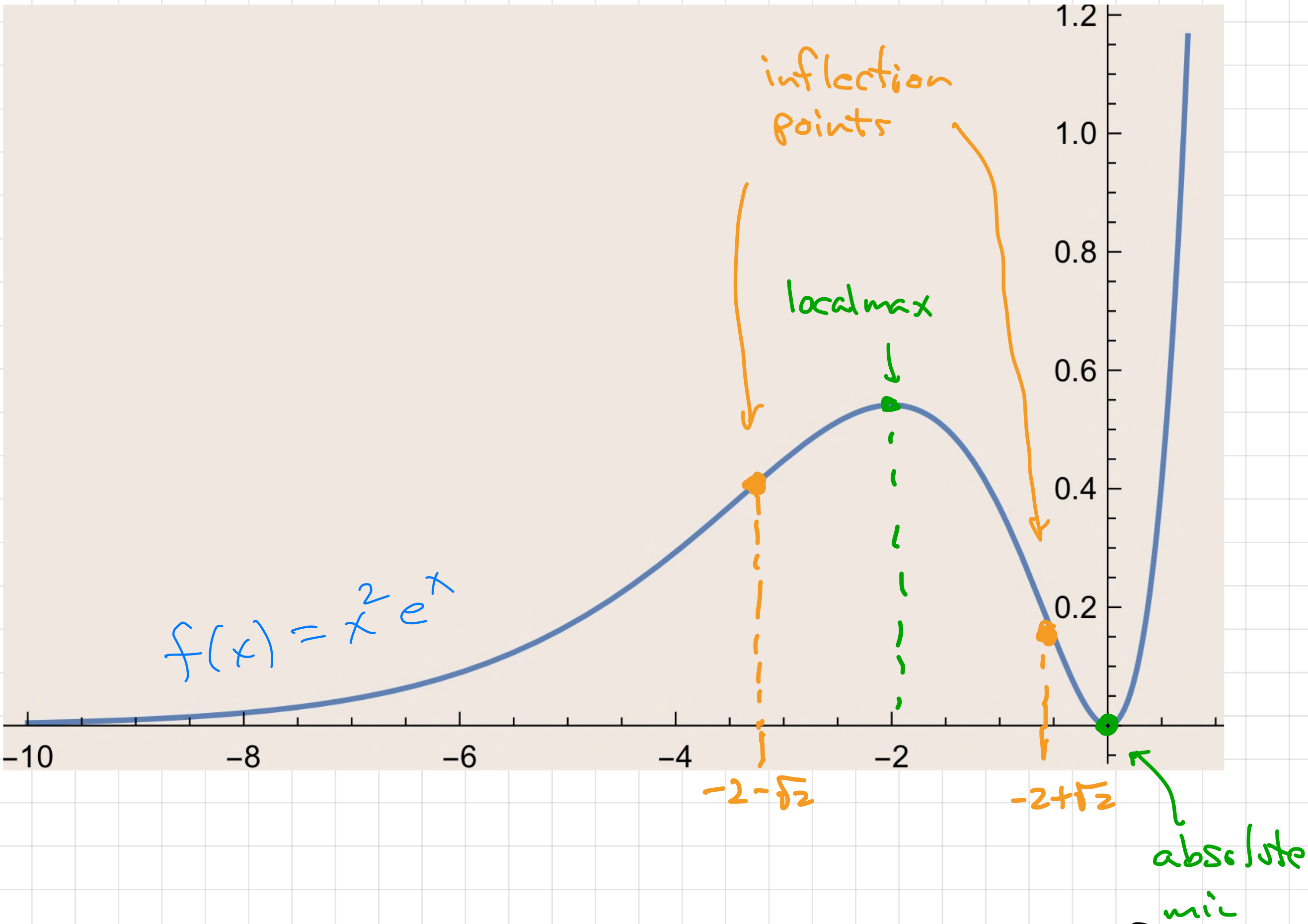
$f''(x) = 2e^x + 4xe^x + x^2e^x = e^x(2+4x+x^2)$

critical numbers for $f'(x)$: $x=-2-\sqrt{2}\approx-3.4$, $x=-2+\sqrt{2}\approx-.59$

f cu		f cd		f cu
$f''>$	$-2-\sqrt{2}$	$f''<0$	$-2+\sqrt{2}$	$f''>0$

Special Properties: $x^2e^x \geq 0$ for all x means graph of $y=x^2e^x$ is above the x-axis.

Using calculus we know we have chosen a good window for the graph: It encompasses all of the critical points for $f(x)$ and for $f'(x)$:



Notice how this graph is broken into five curve elements:

interval	curve type	shape
$(-\infty, -2-\sqrt{2})$	c.u. and incr.	
$(-2-\sqrt{2}, -2)$	c.d. and incr.	
$(-2, -2+\sqrt{2})$	c.d. and decr	
$(-2+\sqrt{2}, 0)$	c.u. and decr.	
$(0, \infty)$	c.u. and incr.	

$0\cdot\infty$ is an "indeterminate form": What does that mean?

<u>Answer</u>: It means that just knowing that $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \infty$ is not enough information to determine $\lim_{x\to a} f(x)g(x)$.

<u>Example</u> Suppose $f(x) = \frac{1}{7x+2}$ and $g(x) = 3x-15$.

Then

$$\lim_{x\to\infty} f(x) = 0^+$$

$$\lim_{x\to\infty} g(x) = \infty$$

This limit has form $0\cdot\infty$

and

$$\lim_{x\to\infty} f(x)g(x) = \lim_{x\to\infty} \frac{3x-15}{7x+2} = \frac{3}{7}$$

rational function

<u>Conclusion</u> $0\cdot\infty$ must equal $3/7$!!

Do you agree?? (hopefully, no.)

To deal with the Primary Indeterminate Forms:

First see if you can find the limit by rewriting the expression algebraically, or do following:

$0/0$, $\pm\infty/\pm\infty$	use L'Hospital
$0\cdot\infty$	write $f(x)g(x)$ as $\frac{f(x)}{1/g(x)}$ or as $\frac{g(x)}{1/f(x)}$ to get $\frac{0}{0}$ or $\frac{\infty}{\infty}$
1^∞, ∞^0, 0^0	Take logarithms to convert to a previous form
$\infty-\infty$	Multiply $f(x)-g(x)$ by $\frac{f(x)+g(x)}{f(x)+g(x)}$

" $\ln(1^\infty) = \infty\cdot\ln(1) = \infty\cdot 0$ "

to get $\dfrac{f(x)^2-g(x)^2}{f(x)+g(x)}$

Chapter 7 Techniques of Integration.

First Comment: Every rule for differentiation can be written also as a rule for integration but the integration rule is likely to be much more complicated. (Remember: Integration is hard.)

Example $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$

$$\Rightarrow \int \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}\,dx = \frac{f(x)}{g(x)} + C$$

but can we ever recognize when an integral has this form? (not too likely!)

However there is a procedure for turning the product rule into an integration technique — called integration by parts.

Product Rule

u, v are functions of x

$$\frac{d}{dx}\left[u\cdot v\right] = v\frac{du}{dx} + u\frac{dv}{dx}$$

$$\Rightarrow \quad uv + C = \int v\frac{du}{dx}\,dx + \int u\frac{dv}{dx}\,dx$$
$$= \int v\,du + \int u\,dv$$

Now solving for $\int u\,dv$ gives

$$\boxed{\int u\,dv = uv - \int v\,du} \quad \text{IP}$$

comment In this formula we can leave out "$+C$" because both integrals in IP have an implied $+C$. .

IP is the formula for "integration by parts".

To use IP one starts with making a choice for u, and dv. Then calculate $du = u'\,dx$ and $v = \int dv$.

...

$$\boxed{\int u\,dv = uv - \int v\,du}$$ IP

$u = x$

example Use IP to calculate $\int \underbrace{x}_{u} \underbrace{\sec^2(x)\,dx}_{dv}$

For this problem let's choose $u = x$

$$\begin{cases} u = x \\ dv = \sec^2(x)\,dx \end{cases} \Rightarrow \begin{cases} du = dx \\ v = \int \sec^2(x)\,dx = \tan(x) + \cancel{C} \end{cases}$$

$v = \tan(x)$

So

$$\int x \sec^2(x)\,dx = \int u\,dv \overset{IP}{=} uv - \int v\,du$$

$$= \underbrace{x}_{u}\,\underbrace{\tan(x)}_{v} - \int \underbrace{\tan(x)}_{v}\,dx$$

see next page.

$$= x\tan(x) - \ln|\sec(x)| + C$$

Now check the answer!!

use product rul

$$\frac{d}{dx}\left[x\tan(x) - \ln|\sec(x)|\right] =$$

$$\left(\cancel{\tan(x)} + x\sec^2(x)\right) - \frac{1}{\cancel{\sec(x)}} \cdot \cancel{\sec(x)}\,\cancel{\tan(x)}$$

$$= x\sec^2(x)$$

Integrate $\tan(x)$?

substitute

$$\begin{cases} u = \cos(x) \\ du = -\sin(x)\,dx \end{cases}$$

$$\int \tan(x)\,dx = \int \frac{\sin(x)}{\cos(x)}\,dx$$

$$= -\int \frac{-\sin(x)\,dx}{\cos(x)} = -\int \frac{du}{u}$$

$$= -\ln|u| + C = -\ln|\cos x| + C$$

$$= \ln|\sec(x)| + C$$

recall

$$-\ln(a) = (-1)\ln(a)$$

$$= \ln(a^{-1})$$

$$= \ln\left(\frac{1}{a}\right)$$

<u>Problem</u> Calculate $\int \underbrace{\arctan(x)}_{u} \underbrace{dx}_{dv}$.

Use integration by parts:

$$\begin{cases} u = \arctan(x) \\ dv = dx \end{cases} \Rightarrow \begin{cases} du = \frac{1}{1+x^2} dx \\ v = \int dx = x + C \end{cases}$$

Then

$$\int \arctan(x)\, dx = \int u\, dv = uv - \int v\, du$$

$$= x \arctan(x) - \int x \frac{1}{1+x^2} dx$$

$$= x\arctan(x) - \frac{1}{2}\int \frac{1}{w} dw \qquad \begin{cases} w = 1+x^2 \\ dw = 2x\, dx \end{cases}$$

$$= x\arctan(x) - \frac{1}{2}\ln|w| + C$$

$$= x\arctan(x) - \frac{1}{2}\ln(1+x^2) + C$$

$$= x\arctan(x) - \ln\left(\sqrt{1+x^2}\right) + C$$