Problem If $f(x) = \dfrac{x^4 - 3}{(x^2-1)^2(x^2+3x+7)}$ then what is the form of the partial fraction decomposition of $f(x)$? How many constants are there to solve for? 6

denominator $= (x-1)^2(x+1)^2(x^2+3x+7)$

x^2+3x+7 is irreducible but not $x^2-1 = (x-1)(x+1)$

Form $= \dfrac{A}{(x-1)^2} + \dfrac{B}{x-1} + \dfrac{C}{(x+1)^2} + \dfrac{D}{x+1} + \dfrac{Ex+F}{x^2+3x+7}$

note: The number of constants appearing in the partial fraction decomposition of a proper rational function always equals the degree of the polynomial in the denominator.

Partial Fractions:

$$\frac{A}{(ax+b)^i} \quad \text{or} \quad \frac{Ax+B}{(ax^2+bx+c)^i}$$

'linear form'   'quadratic form'

where A, B, a, b, c, i are constants for which i is a positive integer and ax^2+bx+c is an irreducible quadratic.

$\boxed{\text{Theorem: Every proper rational function } f(x) \text{ can be expressed in one and only one way as a sum of partial fractions.}}$

47. $\int x^3(x-1)^{-4}\,dx$

$$\frac{x^3}{(x-1)^4} = \frac{A}{(x-1)^4} + \frac{B}{(x-1)^3} + \frac{C}{(x-1)^2} + \frac{D}{x-1}$$

But substituting $u = x-1 \Rightarrow x = u+1,\ du = dx$ is easier

$$\int \frac{x^3}{(x-1)^4}\,dx = \int \frac{(u+1)^3}{u^4}\,du = \int \frac{u^3+3u^2+3u+1}{u^4}\,du$$

$$= \int \frac{1}{u} + \frac{3}{u^2} + \frac{3}{u^3} + \frac{1}{u^4}\,du = \ldots$$

and this actually shows $A = D = 1$, $B = C = 3$

$$\frac{1}{u} + \frac{3}{u^2} + \frac{3}{u^3} + \frac{1}{u^4} = \frac{x^3}{(x-1)^4}$$

$$= \frac{1}{x-1} + \frac{3}{(x-1)^2} + \frac{3}{(x-1)^3} + \frac{1}{(x-1)^4}$$

How to write $\frac{P(x)}{Q(x)}$ as a sum of partial fractions

when $P(x)/Q(x)$ is reduced.

Step 1 Factor the denominator.

(algebra problem, can be nigh impossible)

Step 2 Determine the form of the sum.

(immediate)

Step 3 Solve for constants.

(straightforward but may be tedious)

Step 4 Ready to calculate the integral.

(linear terms are easier than quadratic)

Goal: Calculate $\int \frac{P(x)}{Q(x)}\,dx$

What if $P(x)/Q(x)$ is not proper?

example: $\displaystyle\int \frac{x^2-6x+13}{x-2}\,dx$ ← not proper

method 1: Write $x^2-6x+13$ as a polynomial in $x-2$:

$$x^2-6x+13 = (x-2)^2 - 2(x-2) + 5$$

then $$\frac{x^2-6x+13}{x-2} = (x-2) - 2 + \frac{5}{x-2}$$ ⇐ now easy to integrate

$$\frac{x^2-6x+13}{x-2} = \frac{(x-2)^2-2(x-2)+5}{x-2} = (x-2)-2+\frac{5}{x-2}$$

method 2: Long division

$$\begin{array}{r|l} & x-4 \\ \hline x-2 & x^2-6x+13 \\ & \underline{x^2-2x} \\ & \quad -4x+13 \\ & \quad \underline{-4x+8} \\ & \qquad 5 \end{array}$$

5 ← remainder

$$= \frac{x^2-6x+13}{x-2} = x-4+\frac{5}{x-2}$$ ⇐ now easy to integrate

44. $\int \sqrt{1+e^x}\,dx$

$u = \sqrt{1+e^x}$

$u^2 = 1 + e^x$

$x = \ln(u^2-1)$

Use a "rationalizing substitution"

$$\begin{cases} u^2 = 1+e^x \\ x = \ln(u^2-1) \\ dx = \dfrac{2u}{u^2-1}\,du \end{cases} \Rightarrow u = \sqrt{1+e^x}$$

note: $u^2 \geq 1$

$$\int \sqrt{1+e^x}\,dx = \int u \frac{2u}{u^2-1}\,du = 2\int \frac{u^2}{u^2-1}\,du$$

$$= 2\int \frac{(u^2-1)+1}{u^2-1}\,du = 2\int 1 + \frac{1}{u^2-1}\,du$$

$$(**) \quad = 2\int 1 + \left(\frac{1/2}{u-1} - \frac{1/2}{u+1}\right)du = 2u + \ln|u-1| - \ln|u+1| + C$$

$$= 2\sqrt{1+e^x} + \ln\left(\sqrt{1+e^x}-1\right) - \ln\left(\sqrt{1+e^x}+1\right) + C$$

$$(**) \quad \frac{1}{u^2-1} = \frac{1}{(u-1)(u+1)} = \frac{A}{u-1} + \frac{B}{u+1} = \frac{(A+B)u + (A-B)}{(u-1)(u+1)}$$

$$\Rightarrow \begin{cases} A+B=0 \\ A-B=1 \end{cases} \Rightarrow B=-A \Rightarrow A=1/2,\ B=-1/2$$

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7.5 EXERCISES

1–82 Evaluate the integral.

1. $\int \frac{\cos x}{1 - \sin x}\,dx$
2. $\int_0^1 (3x+1)^{\sqrt{2}}\,dx$
3. $\int_1^4 \sqrt{y}\ln y\,dy$
4. $\int \frac{\sin^3 x}{\cos x}\,dx$
5. $\int \frac{t}{t^4+2}\,dt$
6. $\int_0^1 \frac{x}{(2x+1)^3}\,dx$
7. $\int_{-1}^1 \frac{e^{\arctan y}}{1+y^2}\,dy$
8. $\int t \sin t \cos t\,dt$
9. $\int_2^4 \frac{x+2}{x^2+3x-4}\,dx$
10. $\int \frac{\cos(1/x)}{x^3}\,dx$
11. $\int \frac{1}{x^3\sqrt{x^2-1}}\,dx$
12. $\int \frac{2x-3}{x^3+3x}\,dx$
13. $\int \sin^5 t \cos^4 t\,dt$
14. $\int \ln(1+x^2)\,dx$
15. $\int x \sec x \tan x\,dx$
16. $\int_0^{\sqrt{2}/2} \frac{x^2}{\sqrt{1-x^2}}\,dx$
17. $\int_0^\pi t\cos^2 t\,dt$
18. $\int_1^4 \frac{e^{\sqrt{t}}}{\sqrt{t}}\,dt$
19. $\int e^{x+e^x}\,dx$
20. $\int e^2\,dx$
21. $\int \arctan\sqrt{x}\,dx$
22. $\int \frac{\ln x}{x\sqrt{1+(\ln x)^2}}\,dx$
23. $\int_0^1 (1+\sqrt{x})^8\,dx$
24. $\int (1+\tan x)^2 \sec x\,dx$
25. $\int_0^1 \frac{1+12t}{1+3t}\,dt$
26. $\int_0^1 \frac{3x^2+1}{x^3+x^2+x+1}\,dx$
27. $\int \frac{dx}{1+e^x}$
28. $\int \sin\sqrt{at}\,dt$
29. $\int \ln\left(x+\sqrt{x^2-1}\right)dx$
30. $\int_{-1}^2 |e^x - 1|\,dx$
31. $\int \sqrt{\frac{1+x}{1-x}}\,dx$
32. $\int_1^3 \frac{e^{3/x}}{x^2}\,dx$
33. $\int \sqrt{3-2x-x^2}\,dx$
34. $\int_{\pi/4}^{\pi/2} \frac{1+4\cot x}{4-\cot x}\,dx$
35. $\int_{-\pi/2}^{\pi/2} \frac{x}{1+\cos^2 x}\,dx$
36. $\int \frac{1+\sin x}{1+\cos x}\,dx$
37. $\int_0^{\pi/4} \tan^3\theta \sec^2\theta\,d\theta$
38. $\int_{\pi/6}^{\pi/3} \frac{\sin\theta\cot\theta}{\sec\theta}\,d\theta$
39. $\int \frac{\sec\theta\tan\theta}{\sec^2\theta - \sec\theta}\,d\theta$
40. $\int_0^\pi \sin 6x \cos 3x\,dx$
41. $\int \theta\tan^2\theta\,d\theta$
42. $\int \frac{\tan^{-1}x}{x^2}\,dx$
43. $\int \frac{\sqrt{x}}{1+x^3}\,dx$
44. $\int \sqrt{1+e^x}\,dx$
45. $\int x^5 e^{-x^3}\,dx$
46. $\int \frac{(x-1)e^x}{x^2}\,dx$
47. $\int x^3(x-1)^{-4}\,dx$
48. $\int_0^1 x\sqrt{2-\sqrt{1-x^2}}\,dx$

Color Key:

- ≡ See 4/14 class notes
- ≡ rational functions (only #25 is not proper)
- ≡ trig substitution
- ≡ rework to get a rational function.

A few more hints:

23. $\int_0^1 (1+\sqrt{x})^8\,dx$ Try $u = 1+\sqrt{x} \Rightarrow x = (u-1)^2$

28. $\int \sin\sqrt{at}\,dt$ Taking $u^2 = at$ converts to $\int u\sin(u)\,du$

29. $\int \ln\left(x+\sqrt{x^2-1}\right)dx$ Do trig substitution then IBP

46. $\int \frac{(x-1)e^x}{x^2}\,dx$ IBP with $u = (x-1)e^x$, $dv = \frac{1}{x^2}\,dx$

<u>Definition</u> If $f(x)$ is defined for all $x \geq a$,

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$$

and if this limit equals $L \neq \pm\infty$ then we say that the improper integral $\int_a^\infty f(x)\,dx$ <u>converges</u> to L.

<u>example</u> $\displaystyle\int_1^\infty \frac{1}{e^x}\,dx = \lim_{b\to\infty} \int_1^b e^{-x}\,dx$

$$\int_1^b \frac{1}{e^x}\,dx = \int_1^b e^{-x}\,dx = -e^{-x}\Big|_1^b = -e^{-b} + e^{-1}$$

$$\lim_{b\to\infty}\left(-e^{-b}+e^{-1}\right) = \lim_{b\to\infty} \overset{\to 0}{-\frac{1}{e^b}} + \overset{\to 1/e}{\frac{1}{e}} = \frac{1}{e}$$

$$\Rightarrow \int_1^\infty \frac{1}{e^x}\,dx = \frac{1}{e} \approx .3678794$$

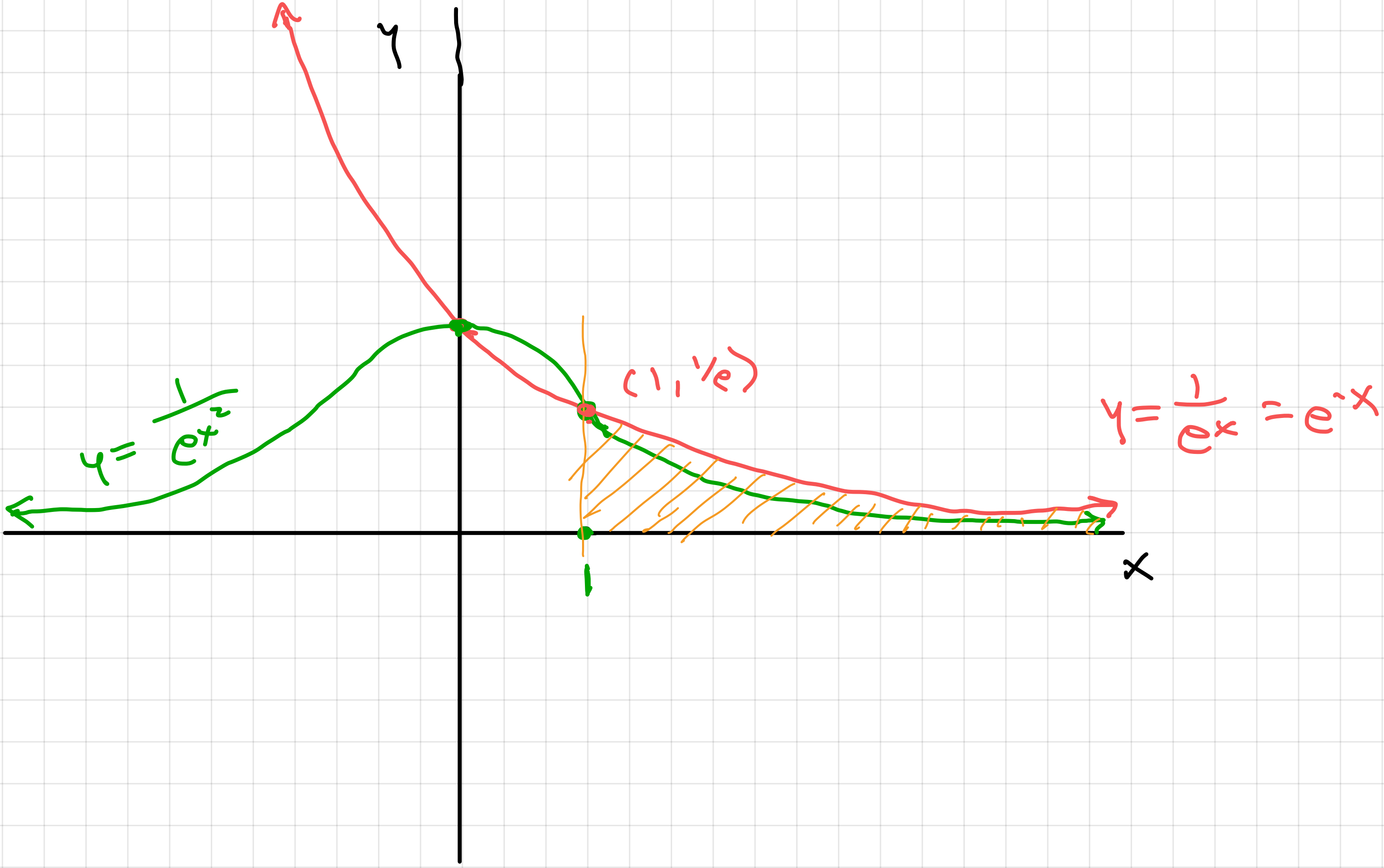
<u>example</u> $\displaystyle\int_1^\infty \frac{1}{e^{x^2}}\,dx = ?$

$$\int_1^b e^{-x^2}\,dx \longleftarrow$$

<u>obstacle</u> $\int e^{-x^2}\,dx$ cannot be worked in closed form.

So calculating $\displaystyle\int_1^\infty \frac{1}{e^{x^2}}\,dx$ precisely looks impossible, but does it converge?

(see next page)



note that: for $x > 1$

$$x^2 \geq x \implies e^{x^2} \geq e^x \implies \frac{1}{e^{x^2}} \leq \frac{1}{e^x}$$

So in Quadrant I, the area under $y = \frac{1}{e^{x^2}}$ for $x \geq 1$ is smaller the area under $y = 1/e^x$.

Since $\int_1^\infty \frac{1}{e^x}\,dx = \frac{1}{e}$ then $\int_1^\infty \frac{1}{e^{x^2}}\,dx < \frac{1}{e}$.

$\implies \int_1^\infty \frac{1}{e^{x^2}}\,dx$ converges.

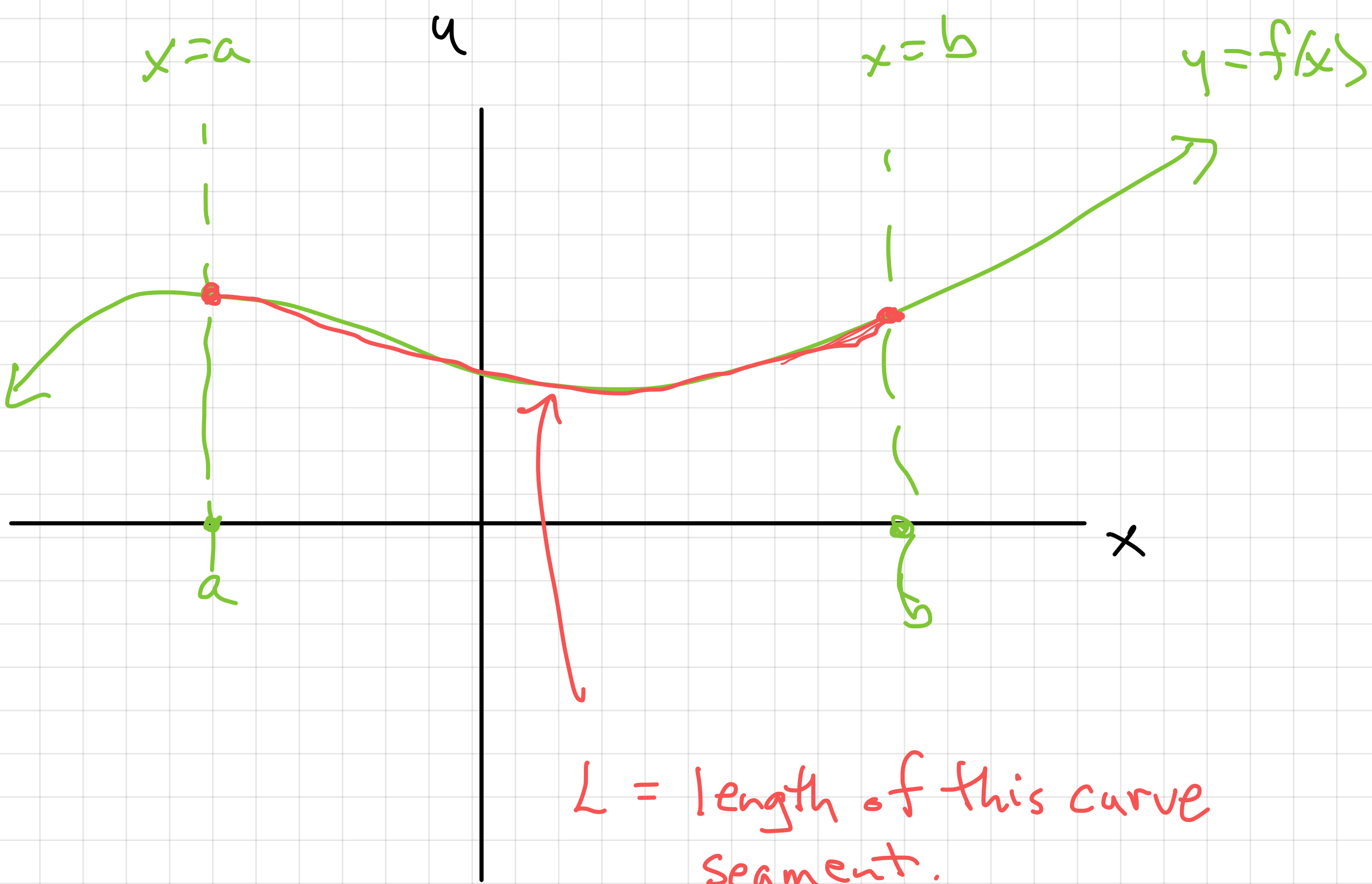
Even though we can't calculate its precise the use of Riemann Sums allows us to numerically approximate this improper integral

$$\int_1^\infty \frac{1}{e^{x^2}}\,dx \approx .1394028$$

Arclength Formula: (section 8.1)

The length L of the curve $y = f(x)$ with $a \le x \le b$ equals

$$L = \int_a^b \sqrt{1 + f'(x)^2}\, dx$$



comment: The arclength formula is very nice and useful in theory but $\int \sqrt{1+f'(x)^2}\, dx$ is often very difficult to evaluate.

example Find the length of $y = 1 + 6x^{3/2}$ for $0 \le x \le 1$.

$$f(x) = 1 + 6x^{3/2}, \quad 0 \le x \le 1$$

$$f'(x) = 6 \tfrac{3}{2} x^{1/2} = 9\sqrt{x}$$

$$\sqrt{1 + (f'(x))^2} = \sqrt{1 + (9\sqrt{x})^2} = \sqrt{1+81x}$$

$$L = \int_0^1 \sqrt{1+81x}\, dx$$

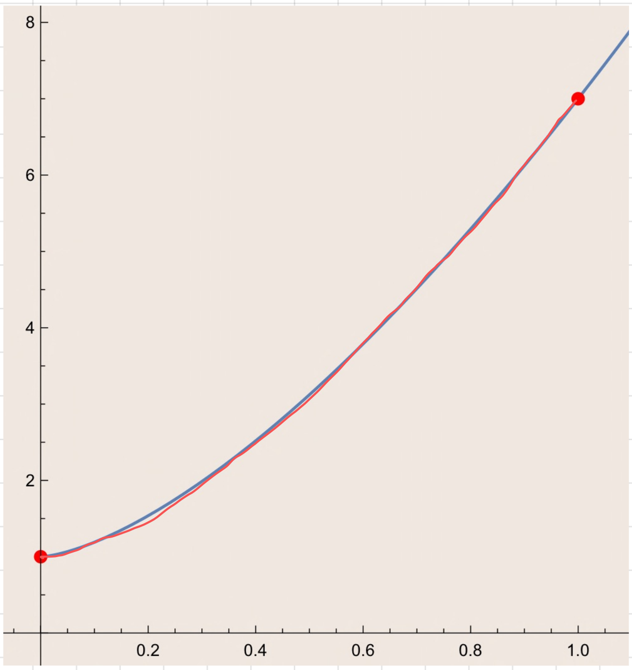
$$\begin{cases} u = 1+81x \\ du = 81\, dx \end{cases}$$

$u(0) = 1$

$u(1) = 82$

$$= \int_1^{82} u^{1/2} \frac{1}{81} dx$$

$$= \frac{1}{81} \frac{2}{3} u^{3/2} \Big|_{u=1}^{82} = \frac{2}{243}\left(82\sqrt{82} - 1\right) \approx 6.1032$$



$y = 1 + 6x^{3/2}$