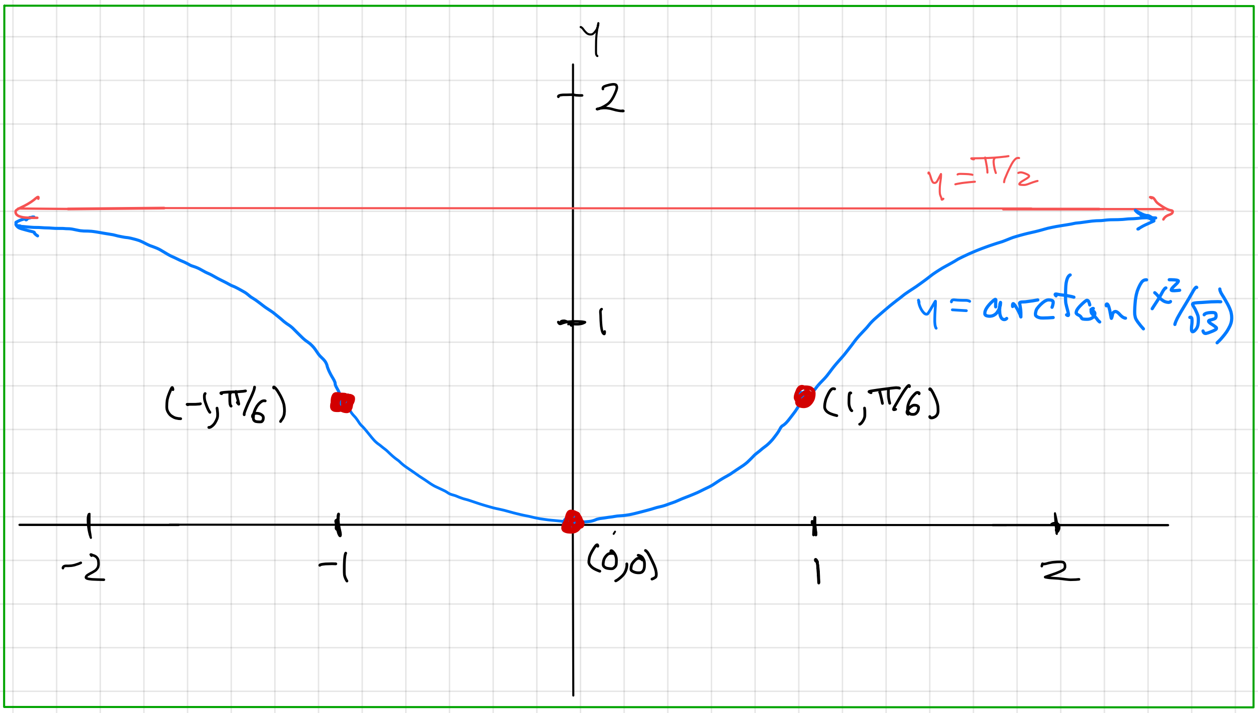
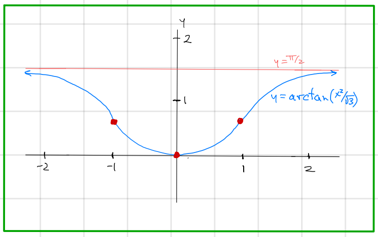
Problem 2 on Exam 3 asked to draw the graph of $f(x) = \arctan(x^2/\sqrt{3})$. A good picture might be:

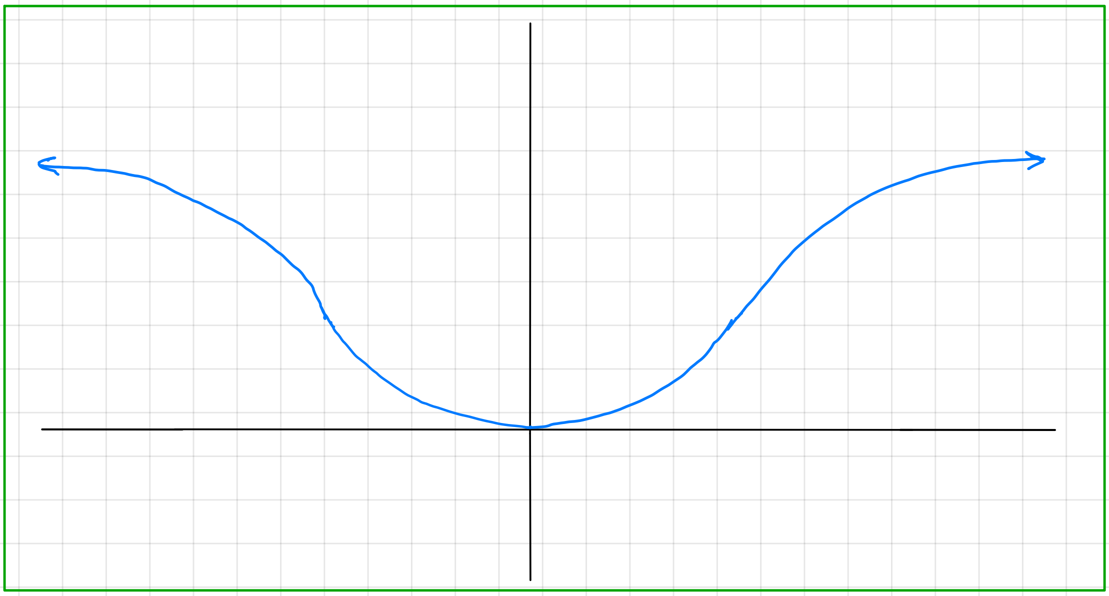


not so good would be:



← way too small!

or:



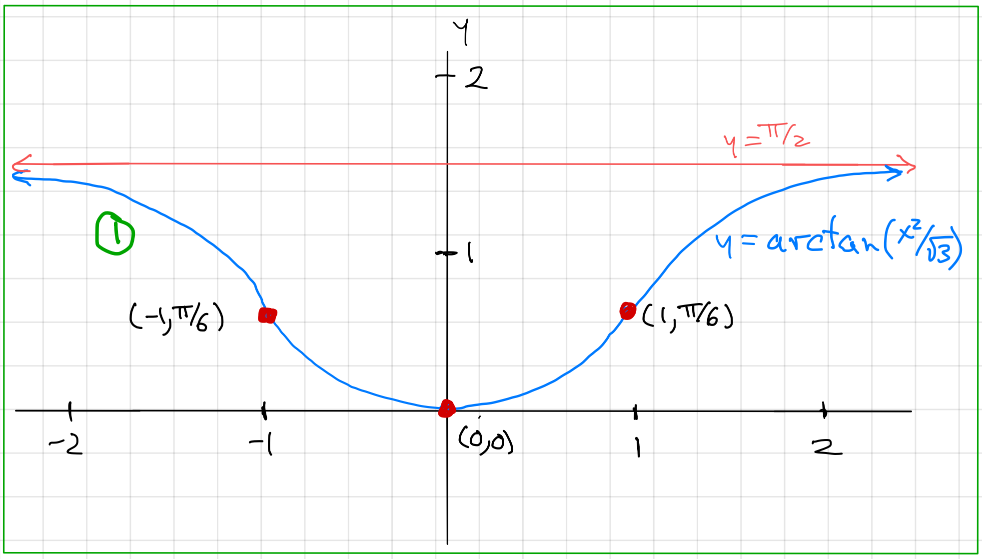
← no labels!

Q: Why so much emphasis on graphs?

A: Being able to construct and analyze skematic diagrams is essential in virtually every area of scientific inquiry. Developing skills to create and use them is critically important.

A good graph helps to understand important properties:

For example: Analyze "curve elements"



The points of inflection and critical points separate the graph into "curve elements".

This graph has four curve elements ....

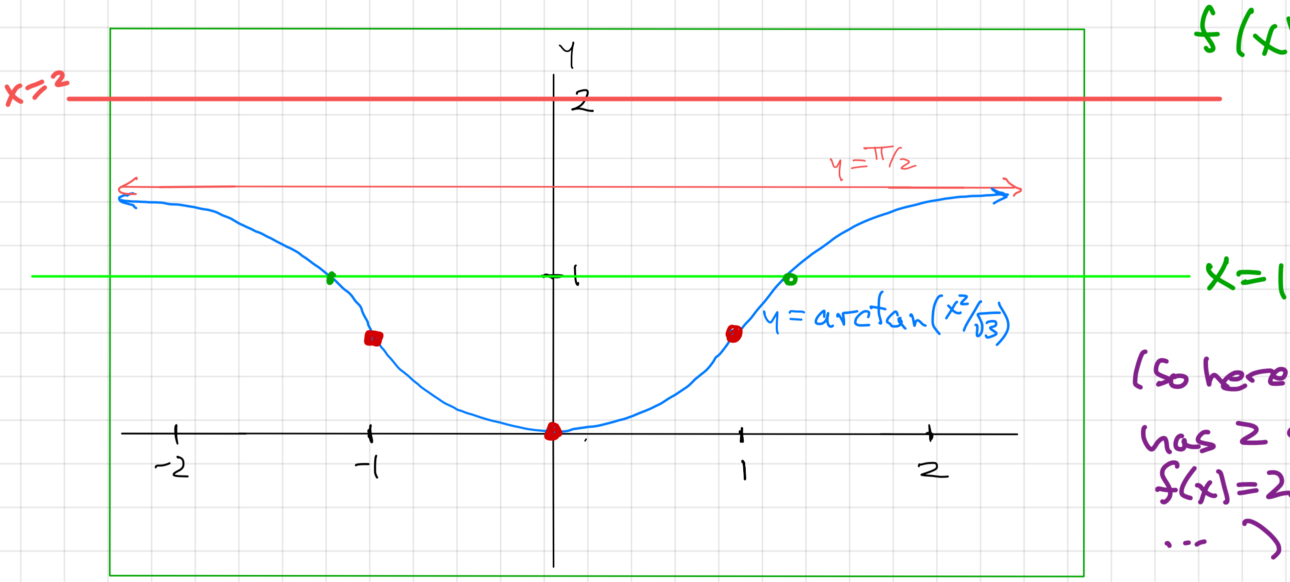
① dec., c.down

etc

(see next page)

or: Get information for solving equations, eg– when does $f(x)=1$?

$f(x)=2$?



(so here $f(x)=1$ has 2 solutions, $f(x)=2$ has none ...)

On the posted Exam 3 solution I also included a key:

x	f(x)	
0	$\arctan(0)=0$	← local min (absolute)
-1	$\arctan(1/\sqrt{3})=\pi/6$	← points of
1	$\arctan(1/\sqrt{3})=\pi/6$	← inflection

The graph of $y=f(x)$ is symmetric across the y-axis because $f(x)$ is an even function.

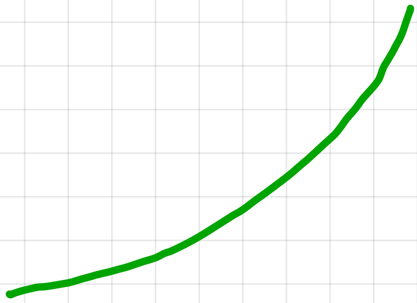
← describes additional information

To include in a graph:

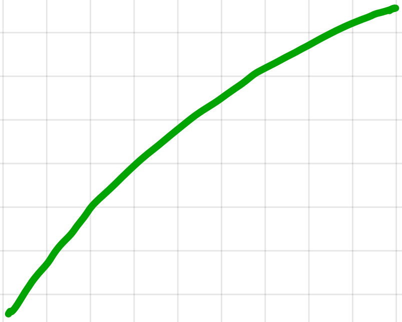
- Clearly label the coordinate axes
- Label all curves with their equation
- Make sure any asymptotes are included
- Mark and give coordinates for all critical points and points of inflection.
  Possibly include x- and y-intercepts as well.
- Adjust aspect ratios as needed to get a useful picture
- Make sure the graph is big and robust enough that it can used to analyze information about the function
- Typically drawing a good graph may take 2 or 3 scratch paper attempts before getting something that looks good.

Four kinds of curve elements:

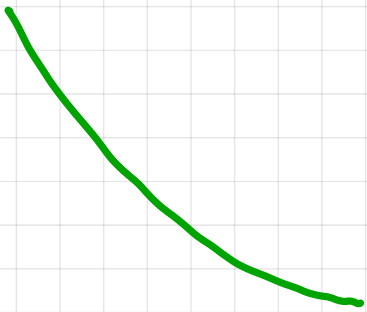
increasing
concave up :



increasing
concave down :



decreasing
concave up :



decreasing
concave down :

Also, a quick note about horizontal asymptotes for

$f(x) = \arctan(x^2/\sqrt{3})$:

We know that

$$\lim_{x\to\infty} \arctan(x) = \pi/2$$

$$\lim_{x\to -\infty} \arctan(x) = -\pi/2$$

and these can be expressed as determinate forms

$$\arctan(\infty) = \pi/2 \quad , \quad \arctan(-\infty) = -\pi/2$$

So

$$\lim_{x\to\infty} \arctan(x^2/\sqrt{3}) = \arctan\left(\lim_{x\to\infty} x^2/\sqrt{3}\right) = \arctan(\infty) = \pi/2$$

$$\lim_{x\to -\infty} \arctan(x^2/\sqrt{3}) = \arctan\left(\lim_{x\to -\infty} x^2/\sqrt{3}\right) = \arctan(\infty) = \pi/2$$

Showing that $y = \pi/2$ is a horizontal asymptote for $y = f(x)$ on both the left and right.

7.5 EXERCISES

Stewart - page 523

1–82 Evaluate the integral.

1. $\int \frac{\cos x}{1 - \sin x}\, dx$
2. $\int_0^1 (3x + 1)^{\sqrt{2}}\, dx$
3. $\int_1^4 \sqrt{y} \ln y\, dy$
4. $\int \frac{\sin^3 x}{\cos x}\, dx$
5. $\int \frac{t}{t^4 + 2}\, dt$
6. $\int_0^1 \frac{x}{(2x + 1)^3}\, dx$
7. $\int_{-1}^1 \frac{e^{\arctan y}}{1 + y^2}\, dy$
8. $\int t \sin t \cos t\, dt$
9. $\int_2^4 \frac{x + 2}{x^2 + 3x - 4}\, dx$
10. $\int \frac{\cos(1/x)}{x^3}\, dx$
11. $\int \frac{1}{x^3\sqrt{x^2 - 1}}\, dx$
12. $\int \frac{2x - 3}{x^3 + 3x}\, dx$
13. $\int \sin^5 t \cos^4 t\, dt$
14. $\int \ln(1 + x^2)\, dx$
15. $\int x \sec x \tan x\, dx$
16. $\int_0^{\sqrt{2}/2} \frac{x^2}{\sqrt{1 - x^2}}\, dx$
17. $\int_0^{\pi} t \cos^2 t\, dt$
18. $\int_1^4 \frac{e^{\sqrt{t}}}{\sqrt{t}}\, dt$
19. $\int e^{x + e^x}\, dx$
20. $\int e^2\, dx$
21. $\int \arctan \sqrt{x}\, dx$
22. $\int \frac{\ln x}{x\sqrt{1 + (\ln x)^2}}\, dx$
23. $\int_0^1 (1 + \sqrt{x})^8\, dx$
24. $\int (1 + \tan x)^2 \sec x\, dx$
25. $\int_0^1 \frac{1 + 12t}{1 + 3t}\, dt$
26. $\int_0^1 \frac{3x^2 + 1}{x^3 + x^2 + x + 1}\, dx$
27. $\int \frac{dx}{1 + e^x}$
28. $\int \sin \sqrt{at}\, dt$
29. $\int \ln\left(x + \sqrt{x^2 - 1}\right) dx$
30. $\int_{-1}^2 | e^x - 1 |\, dx$
31. $\int \sqrt{\frac{1 + x}{1 - x}}\, dx$
32. $\int_1^3 \frac{e^{3/x}}{x^2}\, dx$
33. $\int \sqrt{3 - 2x - x^2}\, dx$
34. $\int_{\pi/4}^{\pi/2} \frac{1 + 4 \cot x}{4 - \cot x}\, dx$
35. $\int_{-\pi/2}^{\pi/2} \frac{x}{1 + \cos^2 x}\, dx$
36. $\int \frac{1 + \sin x}{1 + \cos x}\, dx$
37. $\int_0^{\pi/4} \tan^3\theta \sec^2\theta\, d\theta$
38. $\int_{\pi/6}^{\pi/3} \frac{\sin\theta \cot\theta}{\sec\theta}\, d\theta$
39. $\int \frac{\sec\theta \tan\theta}{\sec^2\theta - \sec\theta}\, d\theta$
40. $\int_0^{\pi} \sin 6x \cos 3x\, dx$
41. $\int \theta \tan^2\theta\, d\theta$
42. $\int \frac{\tan^{-1} x}{x^2}\, dx$
43. $\int \frac{\sqrt{x}}{1 + x^3}\, dx$
44. $\int \sqrt{1 + e^x}\, dx$
45. $\int x^5 e^{-x^3}\, dx$
46. $\int \frac{(x - 1)e^x}{x^2}\, dx$
47. $\int x^3 (x - 1)^{-4}\, dx$
48. $\int_0^1 x\sqrt{2 - \sqrt{1 - x^2}}\, dx$

Color Key:

(purple) = See 4/14 class notes
(orange) = rational functions (only #25 is not proper)
(red) = trig substitution
(blue) = rework to get a rational function.

Even more problems in Section 7.5 :

49. $\displaystyle\int \frac{1}{x\sqrt{4x+1}}\,dx$

50. $\displaystyle\int \frac{1}{x^2\sqrt{4x+1}}\,dx$

51. $\displaystyle\int \frac{1}{x\sqrt{4x^2+1}}\,dx$

52. $\displaystyle\int \frac{dx}{x(x^4+1)}$

53. $\displaystyle\int x^2 \sinh mx\,dx$

54. $\displaystyle\int (x+\sin x)^2\,dx$

55. $\displaystyle\int \frac{dx}{x+x\sqrt{x}}$

56. $\displaystyle\int \frac{dx}{\sqrt{x}+x\sqrt{x}}$

57. $\displaystyle\int x\sqrt[3]{x+c}\,dx$

58. $\displaystyle\int \frac{x\ln x}{\sqrt{x^2-1}}\,dx$

59. $\displaystyle\int \frac{dx}{x^4-16}$

60. $\displaystyle\int \frac{dx}{x^2\sqrt{4x^2-1}}$

61. $\displaystyle\int \frac{d\theta}{1+\cos\theta}$

62. $\displaystyle\int \frac{d\theta}{1+\cos^2\theta}$

63. $\displaystyle\int \sqrt{x}\,e^{\sqrt{x}}\,dx$

64. $\displaystyle\int \frac{1}{\sqrt{\sqrt{x}+1}}\,dx$

65. $\displaystyle\int \frac{\sin 2x}{1+\cos^4 x}\,dx$

66. $\displaystyle\int_{\pi/4}^{\pi/3} \frac{\ln(\tan x)}{\sin x\cos x}\,dx$

67. $\displaystyle\int \frac{1}{\sqrt{x+1}+\sqrt{x}}\,dx$

68. $\displaystyle\int \frac{x^2}{x^6+3x^3+2}\,dx$

69. $\displaystyle\int_1^{\sqrt{3}} \frac{\sqrt{1+x^2}}{x^2}\,dx$

70. $\displaystyle\int \frac{1}{1+2e^x-e^{-x}}\,dx$

71. $\displaystyle\int \frac{e^{2x}}{1+e^x}\,dx$

72. $\displaystyle\int \frac{\ln(x+1)}{x^2}\,dx$

73. $\displaystyle\int \frac{x+\arcsin x}{\sqrt{1-x^2}}\,dx$

74. $\displaystyle\int \frac{4^x+10^x}{2^x}\,dx$

75. $\displaystyle\int \frac{dx}{x\ln x - x}$

76. $\displaystyle\int \frac{x^2}{\sqrt{x^2+1}}\,dx$

77. $\displaystyle\int \frac{xe^x}{\sqrt{1+e^x}}\,dx$

78. $\displaystyle\int \frac{1+\sin x}{1-\sin x}\,dx$

79. $\displaystyle\int x\sin^2 x\cos x\,dx$

80. $\displaystyle\int \frac{\sec x\cos 2x}{\sin x+\sec x}\,dx$

81. $\displaystyle\int \sqrt{1-\sin x}\,dx$

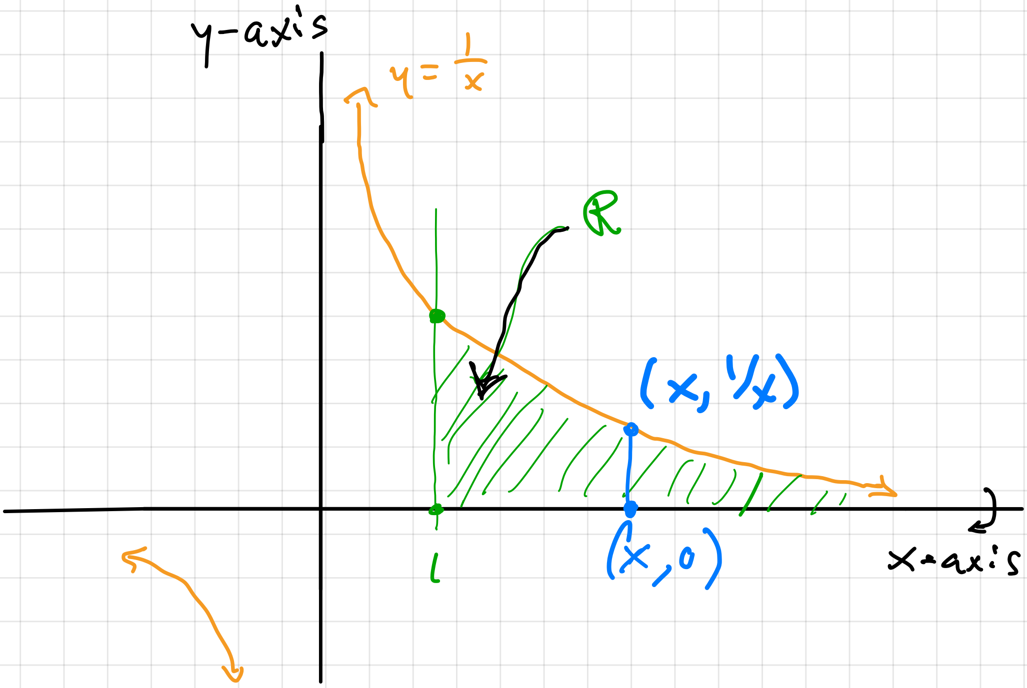
82. $\displaystyle\int \frac{\sin x\cos x}{\sin^4 x+\cos^4 x}\,dx$

Now back to 'improper integrals, Section 7.8 $\rightarrow$

To show:

<u>example</u> The region R below $y = 1/x$ and above $y=0$ for $x \geq 1$ has infinite area but if it is rotated around the x-axis the resulting solid S has finite volume.

picture:

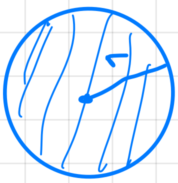


$$\boxed{R : \begin{cases} 1 \leq x < \infty \\ 0 \leq y \leq 1/x \end{cases}}$$

<u>Area of R</u> $= \int_1^\infty \frac{1}{x} - 0 \, dx = \int_1^\infty \frac{1}{x} \, dx = \longrightarrow$

<u>Volume of S</u> $= \int_1^\infty \frac{\pi}{x^2} \, dx$

<u>Disk at x:</u>



$r = \text{radius} = 1/x$

$\text{Area(disk)} = \pi / x^2$

Now calculate the improper integrals $\rightarrow$

"divergent" improper integral

① $$\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx = \lim_{b\to\infty} \ln|x|\Big|_1^b$$

$$= \lim_{b\to\infty} \ln(b) - \cancel{\ln(1)}^{\,0} = \lim_{b\to\infty} \ln(b) = \infty$$

② $$\int_1^\infty \frac{\pi}{x^2}\,dx = \lim_{b\to\infty} \pi\left(\int_1^b x^{-2}\,dx\right)$$

$$= \lim_{b\to\infty} \pi\left((-x^{-1})\Big|_1^b\right) = \lim_{b\to\infty} \left(\cancel{-\frac{\pi}{b}}^{\,0} + \pi\right)$$

$$= \pi$$

"convergent" integral

Other types of improper integrals:

$\ln(0) = \text{DNE}$

<u>example</u> $\int_0^1 \ln(x)\,dx$

Since $\ln(x)$ is only defined for $x>0$ this definite integral is undefined but we can view it as an <u>improper integral</u>:

$$\int_0^1 \ln(x)\,dx = \lim_{\substack{a\to 0+\\ a>0}} \int_a^1 \ln(x)\,dx$$

$$\overset{①}{=} \lim_{a\to 0+} -1 + \underbrace{a}_{0} - \overbrace{\cancel{a\ln a}}^{0}$$

$$\overset{②}{=} -1$$

$\boxed{\text{so this is a convergent integral.}}$

area interpretation? $\longrightarrow$

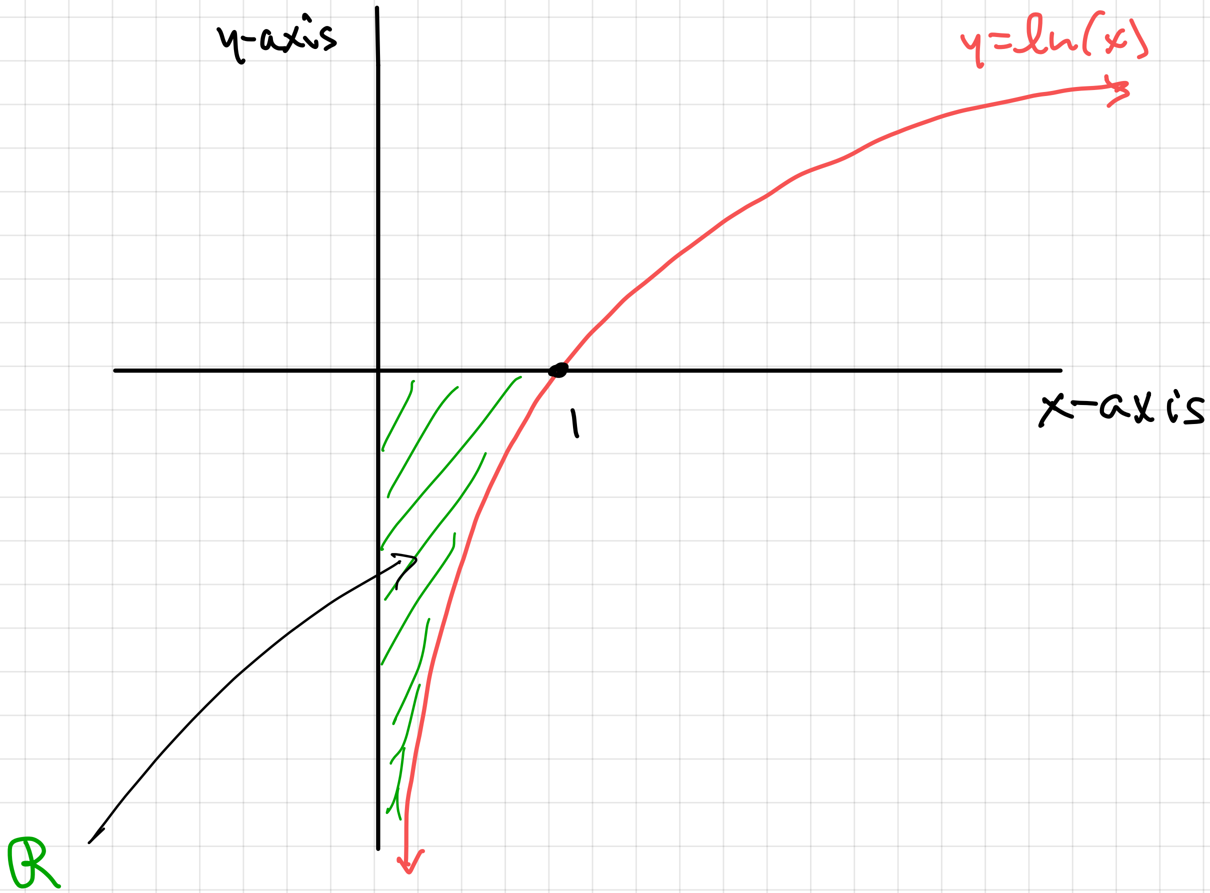
Aside:

① $\int_a^1 \ln x\,dx = x\ln x - x\Big|_a^1 = (1\cdot \overbrace{\cancel{\ln(1)}}^{0} - 1) - (a\ln a - a)$

$= -1 + a - a\ln a$

② $\lim_{a\to 0+} a\ln a$ ← $0\cdot\infty$ form $= \lim_{a\to 0+} \dfrac{\ln a}{1/a}$ ← $0/0$ form

$$\overset{L'H}{=} \lim_{a\to 0+} \frac{1/a}{-1/a^2} = \lim_{a\to 0+} -a = 0$$



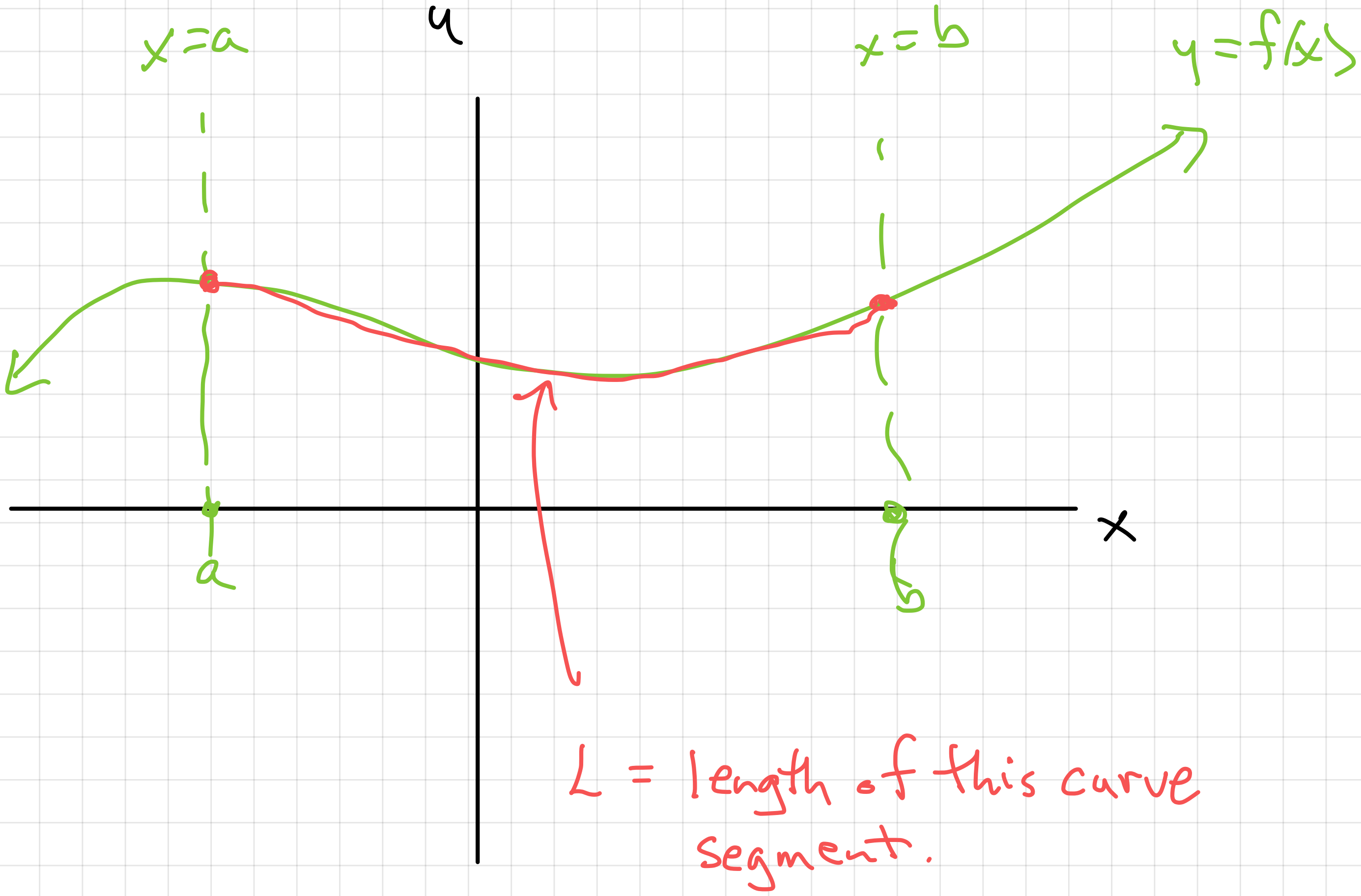
The region R : $\begin{cases} 0 < x \leq 1 \\ \ln x \leq y \leq 0 \end{cases}$ has area

$$\text{area}(R) = \int_0^1 0 - \ln(x)\,dx = -\int_0^1 \ln(x)\,dx$$

$$= -(-1) = 1$$

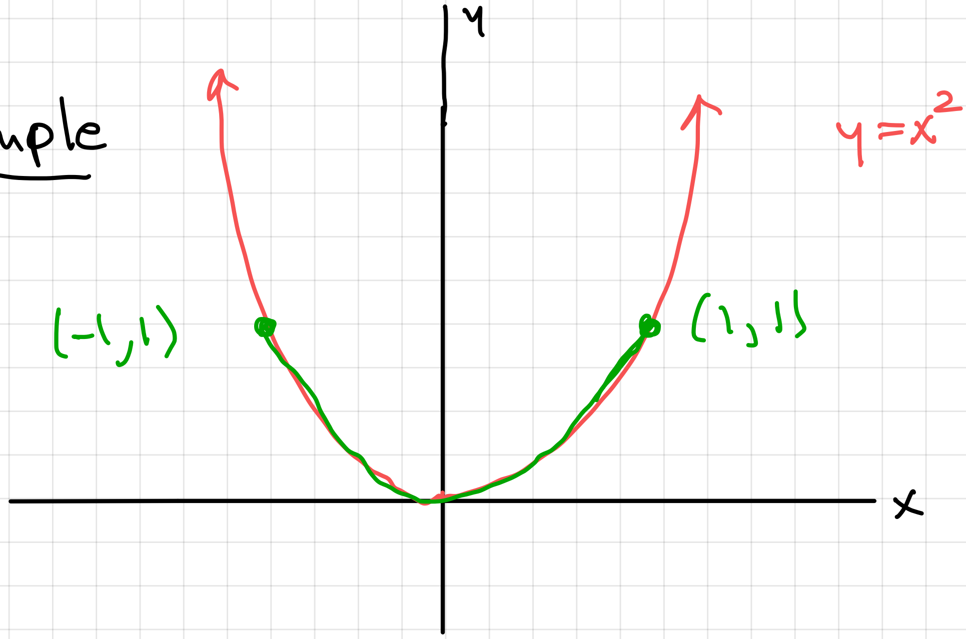
Arclength Formula: (section 8.1)

The length L of the curve $y = f(x)$ with $a \le x \le b$ equals $$L = \int_a^b \sqrt{1 + f'(x)^2}\, dx$$



comment The arclength formula is very nice and useful in theory but $\int \sqrt{1+f'(x)^2}\, dx$ is often very difficult to evaluate.

example



Find the length of segment of the parabola $y=x^2$ from $(-1,1)$ to $(1,1)$.