Problem Review Session May 6

69. $\displaystyle\int_1^{\sqrt{3}} \frac{\sqrt{1+x^2}}{x^2}\,dx$

$x^2+a^2 \rightsquigarrow x = a\tan\theta$

$x^2+1 \rightsquigarrow x = \tan\theta$

substitute: $\begin{cases} x = \tan\theta \\ dx = \sec^2\theta\, d\theta \end{cases}$

$x^2+1 = \tan^2\theta + 1$

$x^2+1 = \sec^2\theta$

$$\int \frac{\sqrt{1+x^2}}{x^2}\,dx = \int \frac{\sec\theta}{\tan^2\theta}\sec^2\theta\,d\theta$$

$$= \int \frac{\sec^3\theta}{\tan^2\theta}\,d\theta = \int \frac{1}{\sin^2\theta\cos\theta}\,d\theta$$

$$= \int \frac{\cos\theta}{\sin^2\theta\cos^2\theta}\,d\theta = \int \frac{1}{\sin^2(\theta)(1-\sin^2\theta)}\cos\theta\,d\theta$$

$\begin{cases} u = \sin\theta \\ du = \cos\theta\, d\theta \end{cases}$

$$= \int \frac{1}{u^2(1-u^2)}\,du = \ldots$$

now use partial fractions ....

$$\frac{\sec^3\theta}{\tan^2\theta} = \frac{\frac{1}{\cos^3\theta}}{\frac{\sin^2\theta}{\cos^2\theta}}\,\frac{\cos^3\theta}{\cos^3\theta} = \frac{1}{\sin^2\theta\cos\theta}$$

Trig Substitution

$$a^2 - x^2 \rightsquigarrow x = a\sin\theta$$

$$\int \sqrt{4-x^2}\, dx$$

$$\begin{cases} x = 2\sin\theta \\ dx = 2\cos\theta\, d\theta \end{cases}$$

$$4 - x^2 = 4 - 4\sin^2\theta$$
$$= 4(1-\sin^2\theta)$$
$$= 4\cos^2\theta$$

$$\sin^2\theta + \cos^2\theta = 1$$

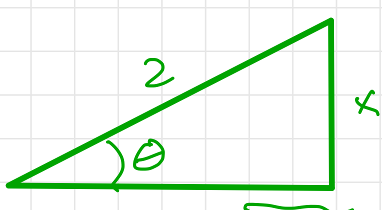
$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$\cos 2\theta = 2\cos^2\theta - 1$$
$$\cos^2\theta = \tfrac{1}{2}(1+\cos 2\theta)$$
$$\sin^2\theta = \tfrac{1}{2}(1-\cos 2\theta)$$

$$= \int 2\cos\theta \cdot 2\cos\theta\, d\theta$$
$$= 4\int \cos^2\theta\, d\theta$$
$$= 4\int \tfrac{1}{2}(1+\cos 2\theta)\, d\theta$$
$$= 2\int 1 + \cos 2\theta\, d\theta$$
$$= 2\left(\theta + \tfrac{1}{2}\sin 2\theta\right) + C$$
$$= 2\left(\sin^{-1}(x/2) + \tfrac{1}{4}x\sqrt{4-x^2}\right) + C$$
$$= 2\sin^{-1}(x/2) + \tfrac{1}{2}x\sqrt{4-x^2} + C$$

$$x = 2\sin\theta \longrightarrow \frac{x}{2} = \sin\theta = \frac{\text{opp}}{\text{hyp}}$$
$$\theta = \sin^{-1}(x/2)$$

$$\sin 2\theta = 2\sin\theta\cos\theta = x\,\frac{\sqrt{4-x^2}}{2}$$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{\sqrt{4-x^2}}{2}$$



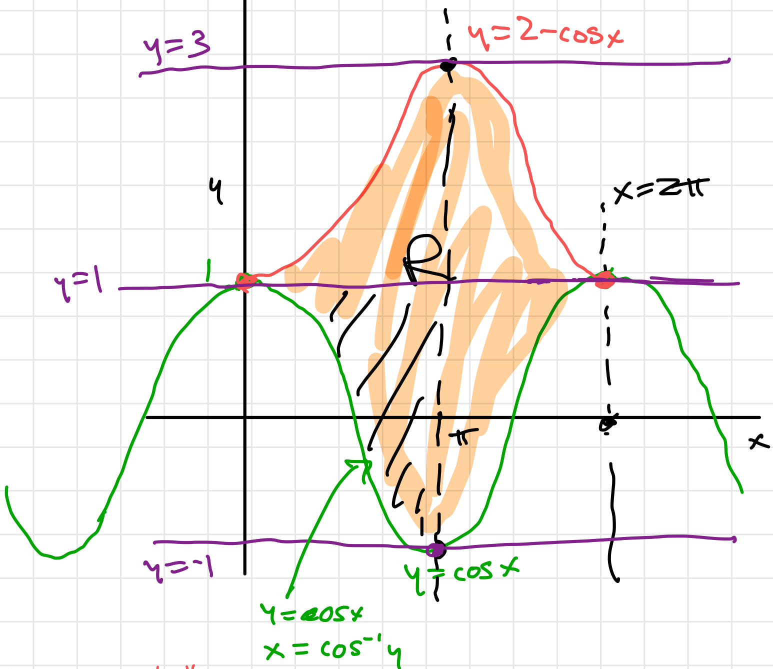
$$a = \sqrt{4-x^2}$$

$$\int \cos 2\theta\, d\theta$$
$$= \tfrac{1}{2}\int \cos u\, du$$
$$= \tfrac{1}{2}\sin(2\theta) + C$$

$$a^2 + x^2 = 4$$
$$a^2 = 4 - x^2 \qquad a = \sqrt{4-x^2}$$

Area Problem:

Volume



$$\begin{cases} y=\cos x \\ y = 2-\cos x \\ 0 \le x \le 2\pi \end{cases}$$

Find the area of region R between these 2 curves.

leftside right

$$R: \begin{cases} 0 \le x \le 2\pi \\ \cos x \le y \le 2-\cos x \end{cases}$$

bottom top

Type I region

top bottom

$$\text{Area}(R) = \int_0^{2\pi} (2-\cos x) - (\cos x)\, dx$$

$$= 2x - 2\sin x \Big|_0^{2\pi} = 4\pi$$

What if we integrate with respect to y instead?

i.e. — Can we view R as a Type II?

Yes — sort of.

Type II regions have horizontal lines on top and bottom

$$\text{Type II} \begin{cases} -1 \le y \le 3 \\ \quad \le x \le \end{cases}$$

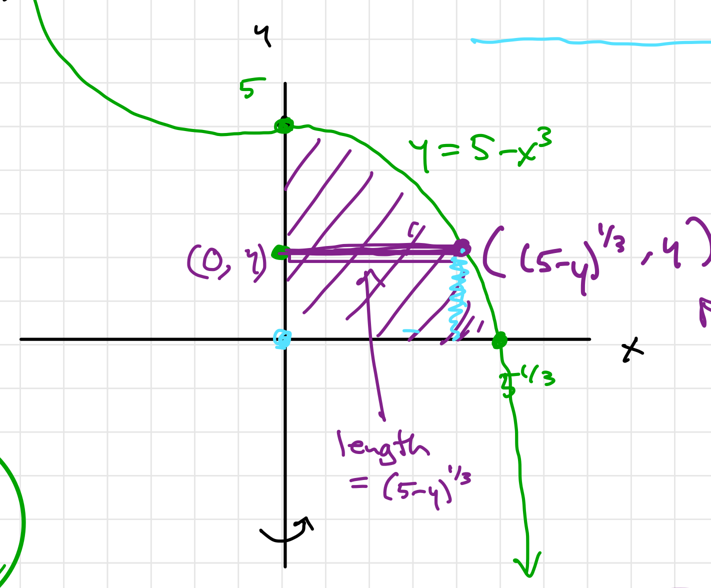
left right

This problem is not really amenable to using dy unless maybe you split R into 4 pieces with the same area (as shown)

volume

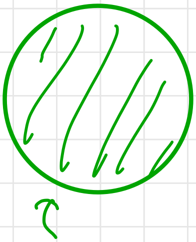
$f(x) = 5 - x^3 \qquad 0 \le x \le 5^{1/3}$

The shaded region is rotated around y-axis. Find the volume of the resulting solid



disk method

dx? or dy?



disk at y has radius $= (5-y)^{1/3}$

$$y = 5 - x^3$$
$$x^3 = 5 - y$$
$$x = (5-y)^{1/3}$$

$$\text{Area(disk)} = \pi (5-y)^{2/3}$$

$$\text{Volume solid} = \int_0^5 \pi (5-y)^{2/3}\, dy = \text{- - -}$$

shell method dx

$$\text{area(shell)} = 2\pi x (5 - x^3)$$

$$\text{Volume} = \int_0^{5^{1/3}} 2\pi x (5 - x^3)\, dx = \text{. . .}$$ same