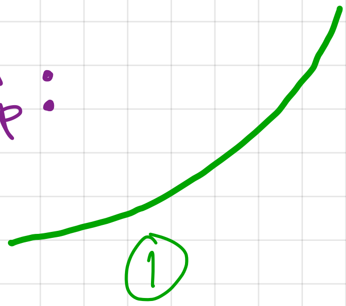
To include in a robust graph:

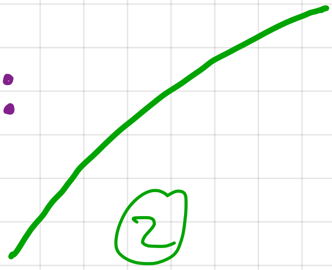
- Clearly label the coordinate axes
- Label all curves with their equation
- Make sure any asymptotes are included
- Mark and give coordinates for all critical points and points of inflection. Possibly include x- and y-intercepts as well.
- Adjust aspect ratios as needed to get a useful picture
- Make sure the graph is big and robust enough that it can used to analyze information about the function
- Typically drawing a good graph may take 2 or 3 scratch paper attempts before getting something that looks good.

Four kinds of curve elements:

increasing
concave up :

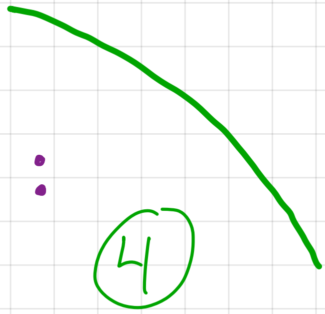


increasing
concave down :

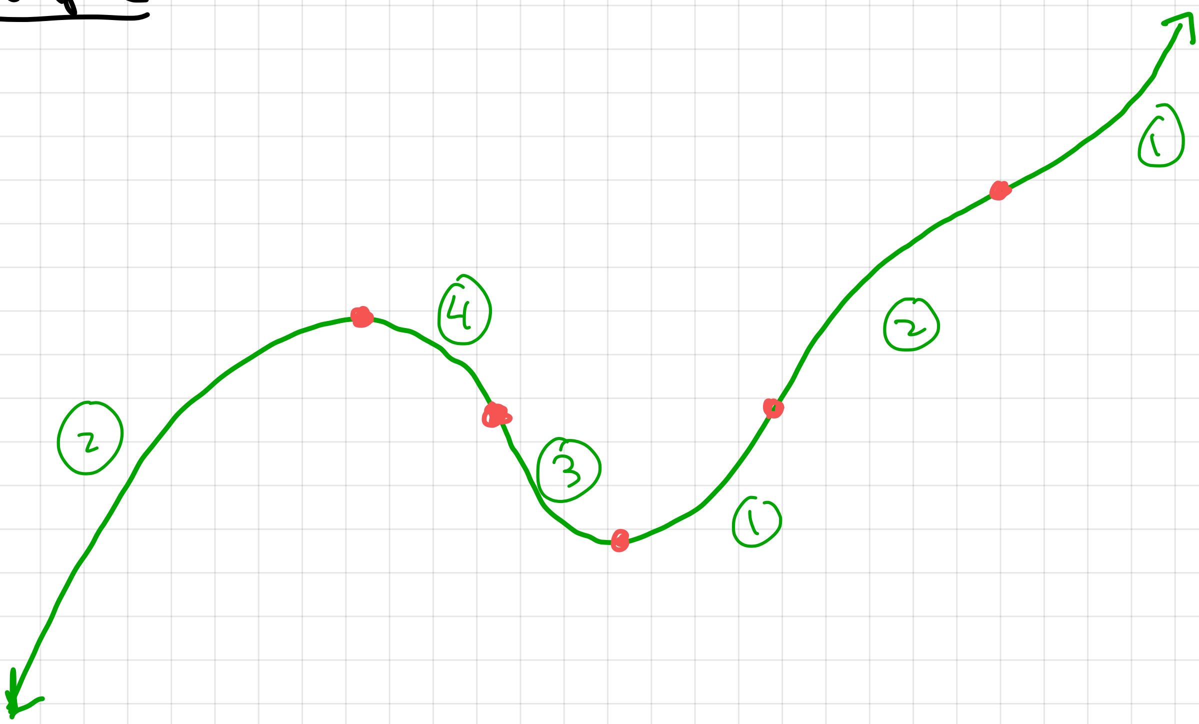


decreasing
concave up :

decreasing
concave down :



example

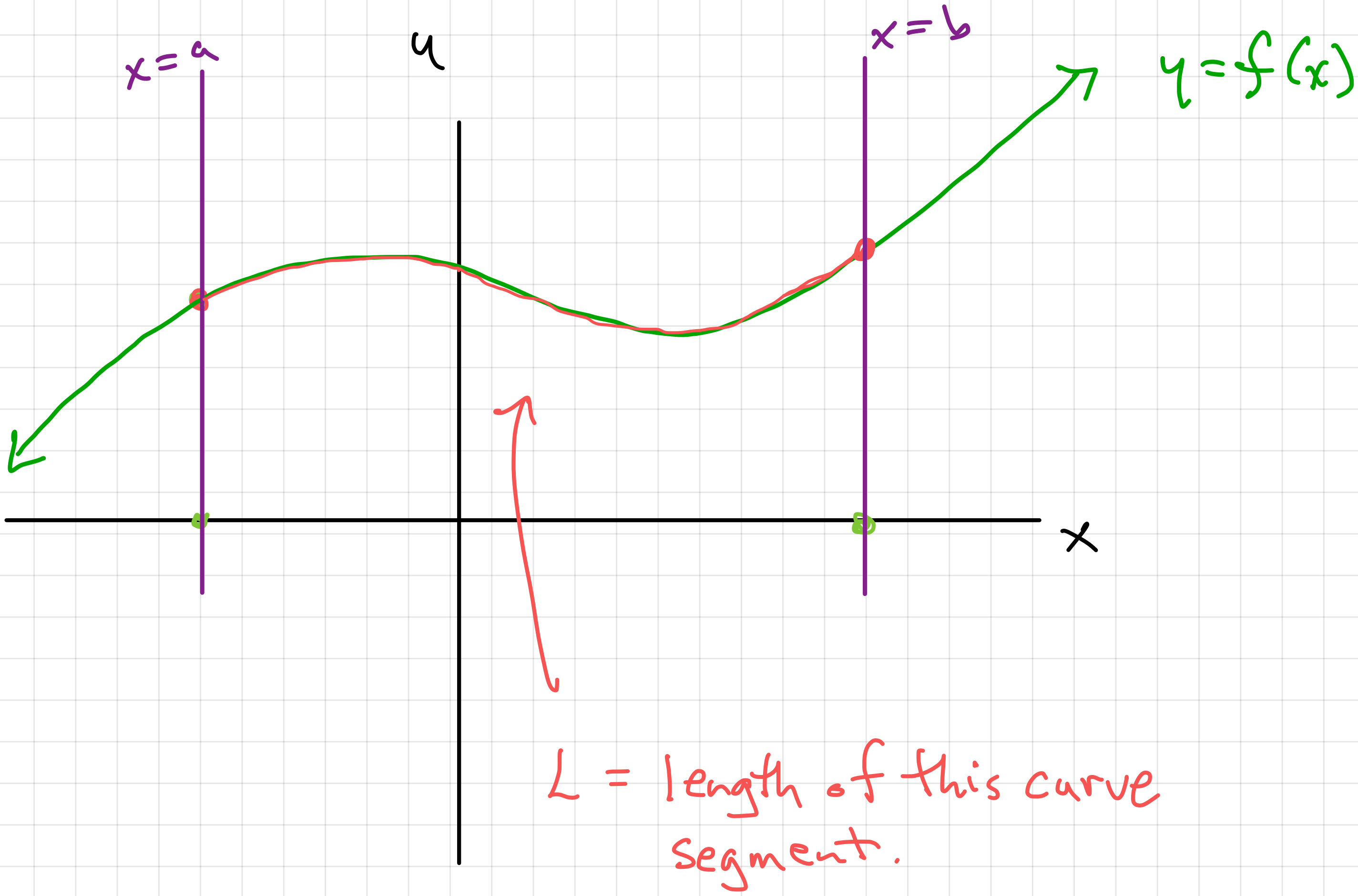


Arclength Formula: (section 8.1)

The length L of the curve $y = f(x)$ with $a \leq x \leq b$ equals

$$L = \int_a^b \sqrt{1 + f'(x)^2}\, dx$$

comes from where?



comment The arclength formula is very nice and useful in theory but $\int_a^b \sqrt{1+f'(x)^2}\, dx$ is often very difficult to evaluate.

Stewart section 8.1, page 589

9–20 Find the exact length of the curve.

9. $y = 1 + 6x^{3/2}, \quad 0 \leqslant x \leqslant 1$

10. $36y^2 = (x^2 - 4)^3, \quad 2 \leqslant x \leqslant 3, \quad y \geqslant 0$

11. $y = \dfrac{x^3}{3} + \dfrac{1}{4x}, \quad 1 \leqslant x \leqslant 2$

12. $x = \dfrac{y^4}{8} + \dfrac{1}{4y^2}, \quad 1 \leqslant y \leqslant 2$

13. $x = \frac{1}{3}\sqrt{y}\,(y - 3), \quad 1 \leqslant y \leqslant 9$

14. $y = \ln(\cos x), \quad 0 \leqslant x \leqslant \pi/3$

15. $y = \ln(\sec x), \quad 0 \leqslant x \leqslant \pi/4$

16. $y = 3 + \frac{1}{2}\cosh 2x, \quad 0 \leqslant x \leqslant 1$

17. $y = \frac{1}{4}x^2 - \frac{1}{2}\ln x, \quad 1 \leqslant x \leqslant 2$

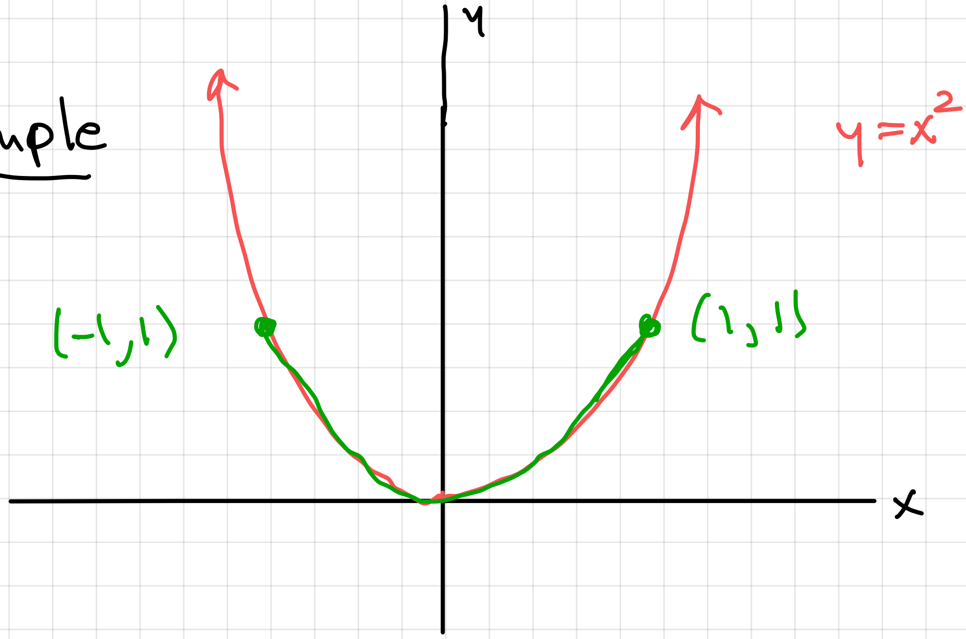
18. $y = \sqrt{x - x^2} + \sin^{-1}\left(\sqrt{x}\right)$

19. $y = \ln(1 - x^2), \quad 0 \leqslant x \leqslant \frac{1}{2}$

20. $y = 1 - e^{-x}, \quad 0 \leqslant x \leqslant 2$

These are some examples where the arclength can be precisely calculated using integration.

example



$f(x)=x^2$

$-1 \leq x \leq 1$

$\quad\| \qquad \|$

$\quad a \qquad b$

Find the length of segment of the parabola $y=x^2$ from $(-1,1)$ to $(1,1)$.

$$f(x)=x^2 \qquad f'(x)=2x$$

$$\Rightarrow \sqrt{1+f'(x)^2} = \sqrt{1+(2x)^2} = \sqrt{1+4x^2}$$

$$L = \int_{-1}^{1} \sqrt{1+4x^2}\,dx$$

$u=2x$

$du=2dx$

$u(1)=2$

$u(-1)=-2$

$$= \frac{1}{2}\int_{-2}^{2} \sqrt{1+u^2}\,du$$

$$\circledast = \frac{1}{2}\left(\frac{u}{2}\sqrt{1+u^2} + \frac{1}{2}\ln\left(u+\sqrt{1+u^2}\right)\right)\Bigg|_{u=-2}^{2}$$

$$= \sqrt{5} + \frac{1}{2}\ln\left(2+\sqrt{5}\right)$$

⊛ This is formula #21 in Stewart backflap.

(see →)

$$\int \sqrt{1+u^2}\, du = ?$$

Make the trig substitution:

$$\begin{cases} u = \tan\theta \\ du = \sec^2\theta\, d\theta \end{cases} \Rightarrow 1+u^2 = 1+\tan^2\theta = \sec^2\theta$$

$$\int \sqrt{1+u^2}\, du = \int \sec\theta \cdot \sec^2\theta\, d\theta = \int \sec^3\theta\, d\theta$$

$$= \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta| + C \quad (**)$$

$$= \frac{1}{2}\sqrt{1+u^2}\cdot u + \frac{1}{2}\ln\left|\sqrt{1+u^2} + u\right| + C$$

(**) Formula 71 in Table of Integrals on back flap of Stewart can be worked using IBP:

take:
$U = \sec\theta \qquad dU = \sec\theta\tan\theta\, d\theta$
$dV = \sec^2\theta\, d\theta \qquad V = \tan\theta$

$$\int \sec^3\theta\, d\theta$$

$$= \int \sec\theta \sec^2\theta\, d\theta$$

$$= \sec\theta\tan\theta - \int \tan\theta \sec\theta \tan\theta\, d\theta$$

$\tan^2\theta = \sec^2\theta - 1$

$$= \sec\theta\tan\theta + \int \sec\theta - \sec^3\theta\, d\theta$$

$$= \sec\theta\tan\theta + \ln|\sec\theta + \tan\theta| - \int \sec^3\theta\, d\theta$$

$\Longrightarrow$

$$\int \sec^3\theta\, d\theta = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta| + C$$

where does arc length formula come from ? . . .

Distance and Integration

Basic Principle:

If an object is in motion over a time interval $[a,b]$ and has speed $s(t)$ at time t then the distance it travels over that interval is $\int_a^b s(t)\,dt$.

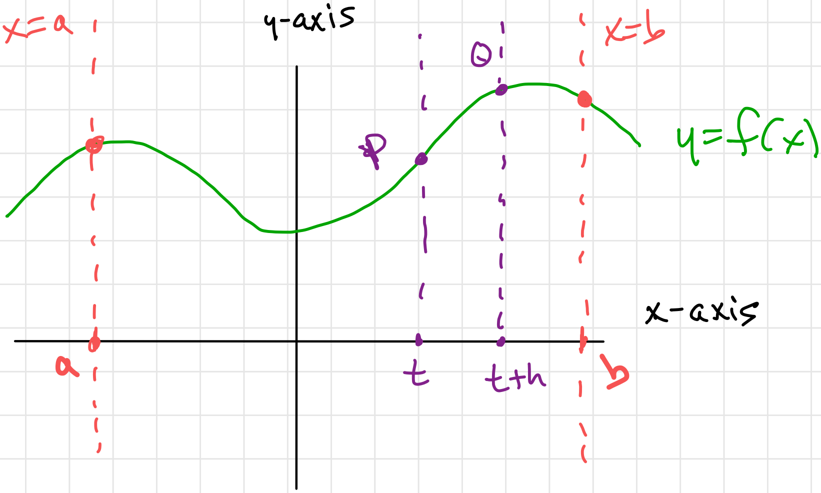
Very Special Case:

Suppose the speed is constant, $s(t)=C$, then

$$\text{distance} = \int_a^b C\,dt = Ct\Big|_{t=a}^{b} = C(b-a) = \text{speed} \times \text{time}$$

speed

elapsed time



Consider an object in motion over the time interval $[a,b]$ so that at time t it is located at the point $P=(t,f(t))$. The average speed over the time interval $[t,t+h]$ is approximately equal to the distance from P to $Q=(t+h,f(t+h))$ divided by h :

dist(P,Q) (use distance formula)

$$\text{average speed from } P \text{ to } Q \approx \frac{\sqrt{(t+h-t)^2+(f(t+h)-f(t))^2}}{h} = \sqrt{\frac{h^2+(f(t+h)-f(t))^2}{h^2}}$$

$$= \sqrt{1+\left(\frac{f(t+h)-f(t)}{h}\right)^2}$$

so (instantaneous) speed at time t is

$$s(t) = \lim_{h\to 0}\sqrt{1+\left(\frac{f(t+h)-f(t)}{h}\right)^2} = \sqrt{1+\left(\lim_{h\to 0}\frac{f(t+h)-f(t)}{h}\right)^2}$$

$$= \sqrt{1+f'(t)^2}$$

<u>Conclude</u> The length L of the curve segment $y=f(x)$, $a\le x\le b$ equals the distance travelled by the object :

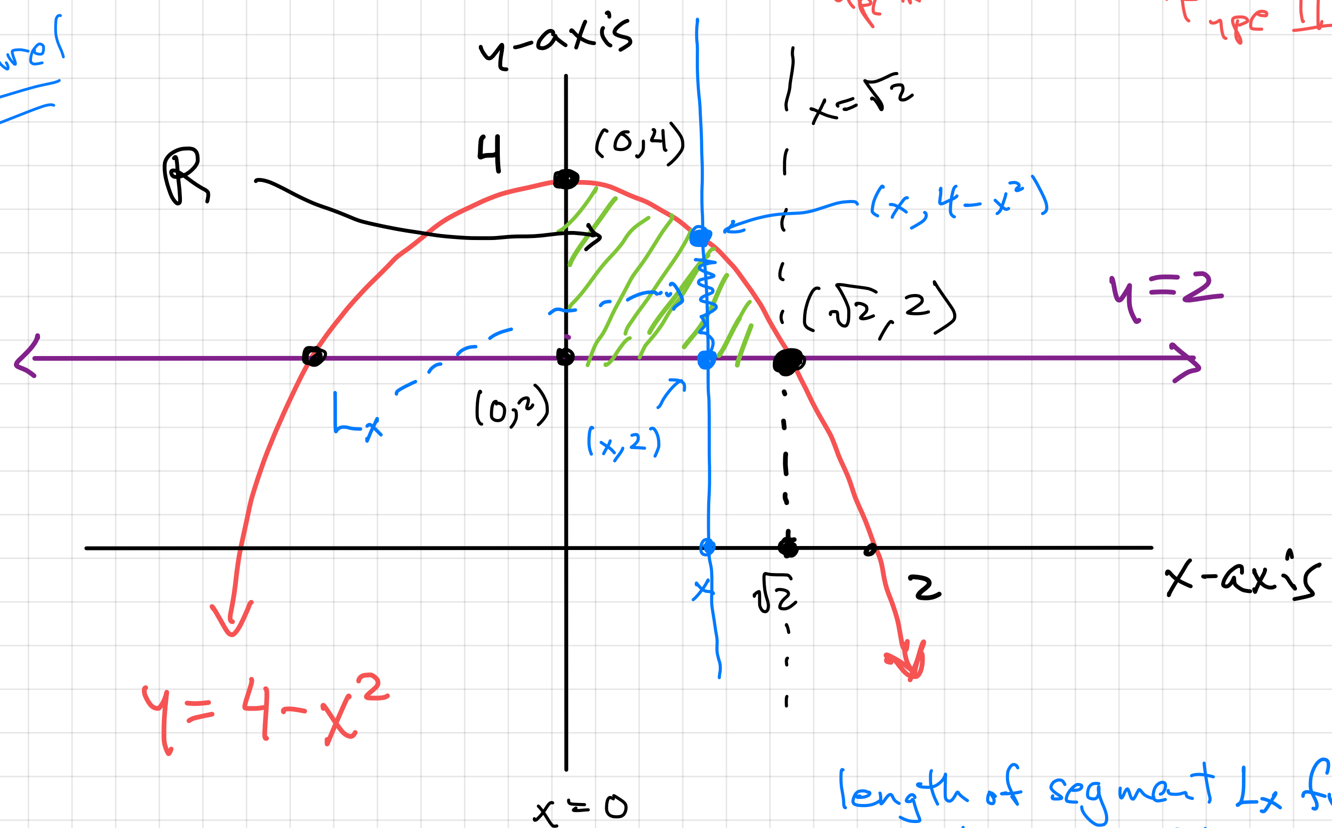
$$L = \int_a^b \underbrace{\sqrt{1+f'(t)^2}}_{=\, s(t)}\, dt$$

Type I and II regions (revisited)

example: Let R be the region in the first quadrant below $y=4-x^2$ and above $y=2$. Calculate area(R) wrt x-axis and wrt y-axis.

Type I (wrt x-axis) ; Type II (wrt y-axis)

figure



length of segment L_x from $(x,2)$ to $(x, 4-x^2)$ is $(4-x^2)-2$ = integrand

(calculate points of intersection of $y=4-x^2$ and $y=2$:

$$4-x^2 = 2 \Rightarrow x^2 = 2 \Rightarrow x = \pm\sqrt{2}$$)

wrt x-axis

R is between the vertical lines $x=0$, $x=\sqrt{2}$

$$R: \begin{cases} \underset{\text{vert. left}}{0} \le x \le \underset{\text{vert. right}}{\sqrt{2}} \\ \underset{\text{bottom}}{2} \le y \le \underset{\text{top}}{4-x^2} \end{cases}$$

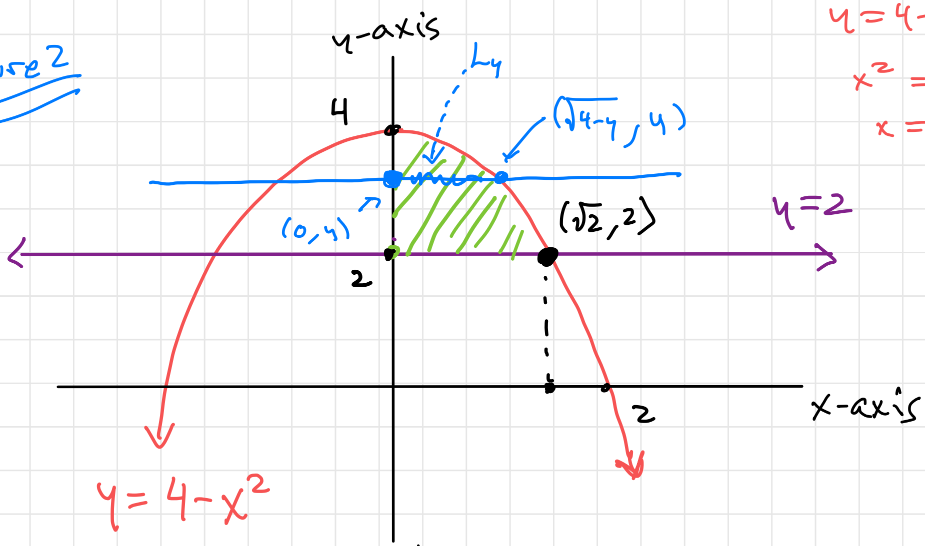
$$\Rightarrow \text{Area}(R) = \int_0^{\sqrt{2}} (4-x^2)-2\,dx = 2x - \tfrac{1}{3}x^3 \Big|_0^{\sqrt{2}} = \tfrac{4}{3}\sqrt{2}$$

$y = 4 - x^2$

$x^2 = 4 - y$

$x = \sqrt{4-y}$

figure 2



wrt y-axis: R is between horizontal lines $y=2$, $y=4$.

$$R = \begin{cases} \overset{\text{horiz bottom}}{2} \le y \le \overset{\text{horiz top}}{4} \\ \underset{\text{left}}{0} \le x \le \underset{\text{right}}{\sqrt{4-y}} \end{cases}$$

$$\begin{cases} u = 4-y & u(2) = 2 \\ du = -dy & u(4) = 0 \end{cases}$$

$$\Rightarrow \text{Area}(R) = \int_2^4 \underbrace{\sqrt{4-y} - 0}_{=\text{length of } L_y}\, dy = -\int_2^0 u^{1/2}\, du$$

$$= -\frac{2}{3} u^{3/2} \Big|_2^0 = \frac{4\sqrt{2}}{3}$$

example: if R is rotated around y-axis what is the volume of the resulting solid?

$$\int_2^4 \pi \left(\sqrt{4-y}\right)^2 dy = \pi \int_2^4 4 - y\, dy = 2\pi$$

$$\int_0^{\sqrt{2}} 2\pi x \left((4-x^2) - 2\right) dx = 2\pi \int_0^{\sqrt{2}} 2x - x^3\, dx = 2\pi$$

comments $\longrightarrow$

To determine the volume we can use:

Disk method:

Take y-axis as reference line (axis of rotation):

For $2 \le y \le 4$ the cross section at y is a disk for which the line segment L_y shown in figure 2 is a radius, and this disk has area:

$$\pi\,(\text{length } L_y)^2 = \pi\left(\sqrt{4-y}\right)^2$$

Integrating from 2 to 4 gives the volume.

or:

shell method:

Take x-axis as reference line ($\perp$ to axis of rotation):

For $0 \le x \le \sqrt{2}$, rotate L_x (as shown in figure 1) about y-axis to get a shell (cylinder) with radius $= x$ and height $=$ length$(L_x) = (4-x^2)-2$. This shell has area $2\pi\,(\text{radius})(\text{height}) = 2\pi x\left((4-x^2)-2\right)$.

Even more problems in Section 7.5:

49. $\int \frac{1}{x\sqrt{4x+1}}\,dx$

50. $\int \frac{1}{x^2\sqrt{4x+1}}\,dx$

51. $\int \frac{1}{x\sqrt{4x^2+1}}\,dx$

52. $\int \frac{dx}{x(x^4+1)}$

53. $\int x^2 \sinh mx\,dx$

54. $\int (x+\sin x)^2\,dx$

55. $\int \frac{dx}{x+x\sqrt{x}}$

56. $\int \frac{dx}{\sqrt{x}+x\sqrt{x}}$

57. $\int x\sqrt[3]{x+c}\,dx$

58. $\int \frac{x\ln x}{\sqrt{x^2-1}}\,dx$

59. $\int \frac{dx}{x^4-16}$

60. $\int \frac{dx}{x^2\sqrt{4x^2-1}}$

61. $\int \frac{d\theta}{1+\cos\theta}$

62. $\int \frac{d\theta}{1+\cos^2\theta}$

63. $\int \sqrt{x}\,e^{\sqrt{x}}\,dx$

64. $\int \frac{1}{\sqrt{\sqrt{x}+1}}\,dx$

65. $\int \frac{\sin 2x}{1+\cos^4 x}\,dx$

66. $\int_{\pi/4}^{\pi/3} \frac{\ln(\tan x)}{\sin x\cos x}\,dx$

67. $\int \frac{1}{\sqrt{x+1}+\sqrt{x}}\,dx$

68. $\int \frac{x^2}{x^6+3x^3+2}\,dx$

69. $\int_1^{\sqrt{3}} \frac{\sqrt{1+x^2}}{x^2}\,dx$

70. $\int \frac{1}{1+2e^x-e^{-x}}\,dx$

71. $\int \frac{e^{2x}}{1+e^x}\,dx$

72. $\int \frac{\ln(x+1)}{x^2}\,dx$

73. $\int \frac{x+\arcsin x}{\sqrt{1-x^2}}\,dx$

74. $\int \frac{4^x+10^x}{2^x}\,dx$

75. $\int \frac{dx}{x\ln x-x}$

76. $\int \frac{x^2}{\sqrt{x^2+1}}\,dx$

77. $\int \frac{xe^x}{\sqrt{1+e^x}}\,dx$

78. $\int \frac{1+\sin x}{1-\sin x}\,dx$

79. $\int x\sin^2 x\cos x\,dx$

80. $\int \frac{\sec x\cos 2x}{\sin x+\sec x}\,dx$

81. $\int \sqrt{1-\sin x}\,dx$

82. $\int \frac{\sin x\cos x}{\sin^4 x+\cos^4 x}\,dx$

A few hints:

#49 (see next page)

#50 $u^2 = 4x+1$

#51 trig substitution $x = \frac{1}{2}\tan\theta$

#52 $u = x^2$ observe that $\frac{1}{x(x^4+1)} = \frac{x}{x^2((x^2)^2+1)}$

#54 expand out

#55 $x = u^2$, $dx = 2u\,du$

#56 $x = u^2$

#57 $x+C = u^3$, $dx = 3u^2\,du$

#69 $u^2 = 1+x^2$ is easier than trig substitution

49. $\displaystyle\int \frac{1}{x\sqrt{4x+1}}\,dx$

Use a "rationalizing substitution":

$$\begin{cases} u^2 = 4x+1 \\ x = \frac{1}{4}(u^2-1) \\ dx = \frac{1}{2}u\,du \end{cases} \Rightarrow u = \sqrt{4x+1}$$

$$= \int \frac{1}{\frac{1}{4}(u^2-1)\cdot u}\,\frac{1}{2}u\,du = \int \frac{2}{u^2-1}\,du$$

$$= \int \frac{1}{u-1} - \frac{1}{u+1}\,du = \ln|u-1| - \ln|u+1| + C$$

$$= \ln\left|\sqrt{4x+1} - 1\right| - \ln\left|\sqrt{4x+1} + 1\right| + C$$

$$= \ln\left|\frac{\sqrt{4x+1} - 1}{\sqrt{4x+1} + 1}\right| + C$$

$$\frac{2}{u^2-1} = \frac{A}{u-1} + \frac{B}{u+1} = \frac{(A+B)u + (A-B)}{(u-1)(u+1)}$$

$$\Rightarrow \begin{cases} A+B=0 \\ A-B=2 \end{cases} \Rightarrow \begin{matrix} A=1 \\ B=-1 \end{matrix}$$