Exam 3
Math 2423
April 19, 2021

Answers

PROBLEM 1. (10 points)

(a) Show how to use integration by parts to calculate $\int (2 - x + 3x^3)x\,dx$ by choosing $u = x$.

(b) Calculate $\int (2 - x + 3x^3)x\,dx$ by a different method and show that your answer agrees with (a).

(a) Take $\begin{cases} u = x \\ dv = (2 - x + 3x^3)\,dx \end{cases} \Rightarrow \begin{matrix} du = dx \\ v = 2x - \frac{x^2}{2} + \frac{3x^4}{4} \end{matrix}$

$$\int x(2 - x + 3x^3)\,dx = \int u\,dv = uv - \int v\,du$$

$$= x\left(2x - \frac{x^2}{2} + \frac{3x^4}{4}\right) - \int 2x - \frac{x^2}{2} + \frac{3}{4}x^4\,dx$$

$$= \left(2x^2 - \frac{x^3}{2} + \frac{3x^5}{4}\right) - \left(x^2 - \frac{x^3}{6} + \frac{3}{20}x^5\right) + C$$

$$= (2-1)\,x^2 + \left(-\frac{1}{2} + \frac{1}{6}\right)x^3 + \left(\frac{3}{4} - \frac{3}{20}\right)x^5 + C$$

$$= x^2 - \frac{x^3}{3} + \frac{3}{5}x^5 + C$$

(b) $$\int (2 - x + 3x^3)x\,dx = \int 2x - x^2 + 3x^4\,dx$$

$$= x^2 - \frac{1}{3}x^3 + \frac{3}{5}x^5 + C$$

= Answer from (a)

PROBLEM 2. (40 points)

In this problem clearly indicate and separate your work for each part (a)-(g).

Let $f(x) = \arctan(x^2/\sqrt{3})$.

(a) Find and simplify formulas for $f'(x)$ and for $f''(x)$.

(b) Determine all of the critical points for $f(x)$.

(c) Determine the intervals on which $f(x)$ is increasing and decreasing.

(d) Determine the intervals of concavity for $f(x)$. (hint: there are two points of inflection.)

(e) Show that $y = f(x)$ has a horizontal asymptote $y = L$ and find L by computing a limit.

(f) Show that $f(x)$ is an even function.

(g) Give a robust and well-labeled sketch of the graph of $y = f(x)$ incorporating all of (a)-(f).

(a) $$f'(x) = \frac{1}{1+(x^2/\sqrt{3})^2} \cdot \frac{2x}{\sqrt{3}} = \frac{2\sqrt{3}\,x}{3+x^4}$$ (used chain rule)

$$f''(x) = \frac{6\sqrt{3}\,(1-x^4)}{(3+x^4)^2}$$ (used quotient rule)

(b) $$\frac{2\sqrt{3}\,x}{3+x^4} = 0 \iff x = 0$$ So (0,0) is only critical pt.

(c)

	$x<0$	0	$x>0$
	f decr		f incr
	$f'<0$	0	$f'>0$

(d) $$f''(x) = 0 \iff (1-x^4) = 0 \iff x = \pm 1$$

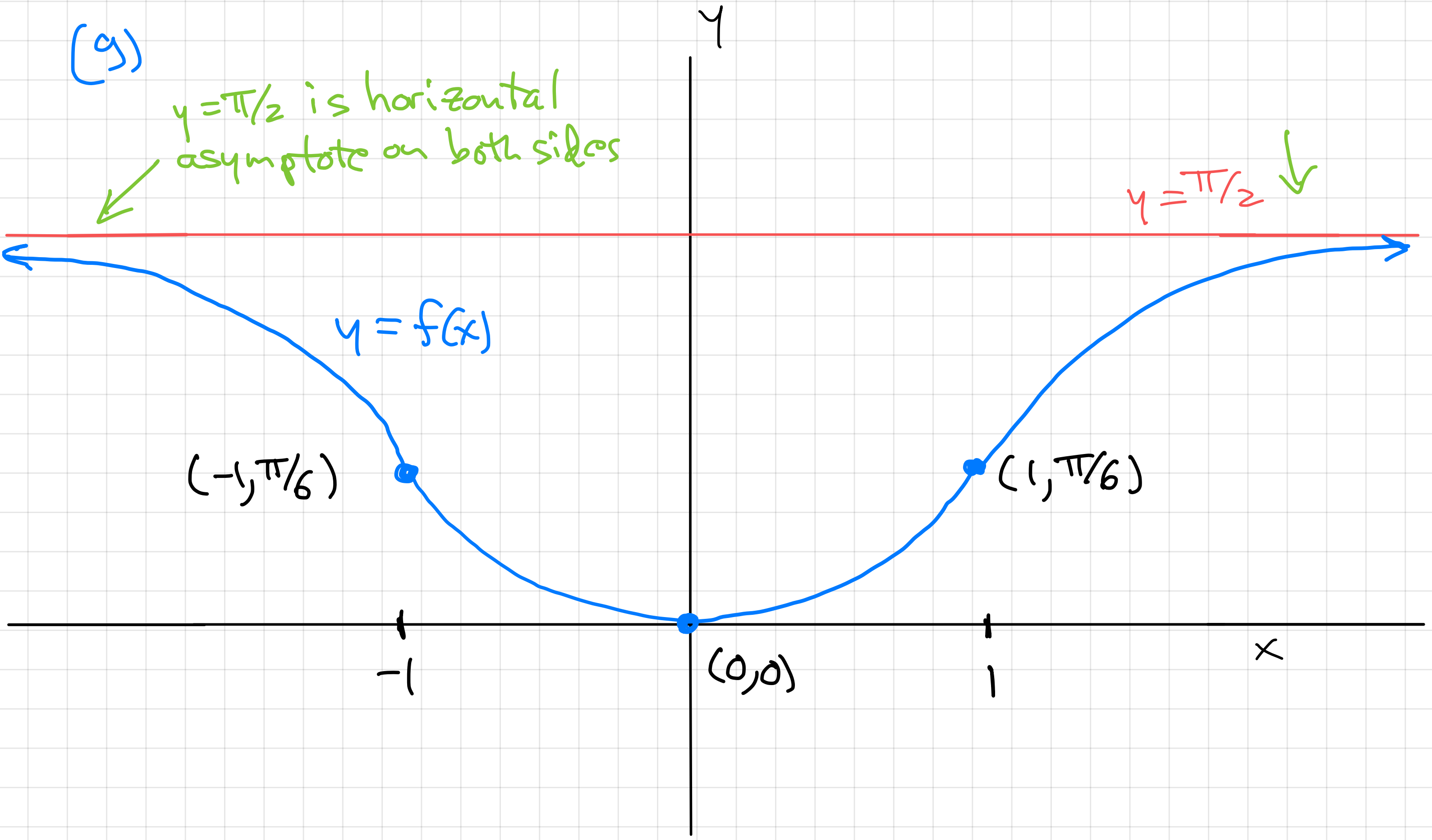
	$x<-1$	-1	$-1<x<1$	1	$x>1$
	f c.d.		f c.u.		f c.d.
	$f''<0$	-1	$f''>0$	1	$f''<0$

(e) $$\lim_{x\to\infty} \arctan(x^2/\sqrt{3}) = \pi/2 = L$$

$$\lim_{x\to-\infty} \arctan(x^2/\sqrt{3}) = \pi/2$$

(f) $f(-x) = \arctan\left(\frac{(-x)^2}{\sqrt{3}}\right) = \arctan\left(\frac{x^2}{\sqrt{3}}\right) = f(x)$

(g)



x	$f(x)$	
0	$\arctan(0) = 0$	← local min (absolute)
-1	$\arctan(1/\sqrt{3}) = \pi/6$	← points of
1	$\arctan(1/\sqrt{3}) = \pi/6$	← inflection

The graph of $y = f(x)$ is symmetric across the y-axis because $f(x)$ is an even function.

PROBLEM 3. Calculate each of the following integrals and write the answer in simplest form. Show your work clearly and indicate any techniques that you use. (40 points)

(a) $\int \sec(\pi x)\, dx$

substitute $\begin{cases} u = \pi x \\ du = \pi\, d\pi \end{cases}$

$\int \sec(\pi x)\, dx = \frac{1}{\pi}\int \sec u\, du = \frac{1}{\pi}\ln|\sec u + \tan u| + C$

$= \frac{1}{\pi}\ln|\sec(\pi x) + \tan(\pi x)| + C$

(b) $\int \tan^3(\theta)\sec(\theta)\, d\theta$ $= \int \tan^2\theta \sec\theta \tan\theta\, d\theta$

$= \int (\sec^2\theta - 1)\sec\theta\tan\theta\, d\theta$ $\begin{cases} u = \sec\theta \\ du = \sec\theta\tan\theta\, d\theta \end{cases}$

$= \int u^2 - 1\, du = \frac{1}{3}u^3 - u + C$

$= \frac{1}{3}\sec^3\theta - \sec\theta + C$ $\tan^2\theta = \sec^2\theta - 1$

(c) $\int \sin^5(x)\, dx$ $= \int (\sin^2(x))^2 \sin(x)\, dx$ $\sin^2 x = 1 - \cos x$

$= \int (1 - \cos^2 x)^2 \sin(x)\, dx$ $\begin{cases} u = \cos x \\ du = -\sin x\, dx \end{cases}$

$= -\int (1 - u^2)^2\, du = \int -1 + 2u^2 - u^4\, du$

$= -u + \frac{2}{3}u^3 - \frac{1}{5}u^5 + C = -\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + C$

(d) $\int \sin^2(x)\cos^2(x)\, dx$

using double angle formulac.

$= \int \frac{1}{2}(1 - \cos(2x))\frac{1}{2}(1 + \cos(2x))\, dx = \frac{1}{4}\int 1 - \cos^2(2x)\, dx$

$= \frac{1}{4}\int 1 - \frac{1}{2}(1 + \cos(4x))\, dx = \frac{1}{8}\int 1 - \cos(4x)\, dx$

$= \frac{1}{8}\left(x - \frac{1}{4}\sin(4x)\right) + C = \frac{x}{8} - \frac{1}{32}\sin(4x) + C$

NOTE: Part (d) was not graded on the exam because of a typo.

(e) $\int_{-3}^{3} \arctan(x)\, dx$ $= 0$

because arctan is an odd function.

(or use integration formula for arctan(x))

PROBLEM 4. A student determines that (10 points)

$$\int \frac{2x^2}{\sqrt{1-x^2}}\,dx = -x\sqrt{1-x^2} + \arcsin(x) + C.$$

Is that answer correct? Explain.

We can check by differentiating the answer:

$$\frac{d}{dx}\left[\left(-x(1-x^2)^{1/2}\right) + \arcsin(x)\right] =$$

$$\left(-(1-x^2)^{1/2} - x\,\tfrac{1}{2}(1-x^2)^{-1/2}(-2x)\right) + \frac{1}{\sqrt{1-x^2}}$$

$$= -\sqrt{1-x^2} + \frac{x^2}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-x^2}}$$

$$= \frac{-\sqrt{1-x^2}\sqrt{1-x^2}}{\sqrt{1-x^2}} + \frac{x^2}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-x^2}}$$

$$= \frac{-(1-x^2) + x^2 + 1}{\sqrt{1-x^2}} = \frac{2x^2}{\sqrt{1-x^2}}$$

So the answer is correct.