#11 section 7.5

$$\int \frac{1}{x^3\sqrt{x^2-1}}\,dx$$

$$\begin{cases} x = \sec\theta \\ x^2-1 = \sec^2\theta - 1 = \tan^2\theta \\ dx = \sec\theta\tan\theta\, d\theta \end{cases}$$

$$= \int \frac{1}{\sec^{\cancel{3}\,2}\theta \cdot \cancel{\tan\theta}}\,\cancel{\sec\theta}\,\cancel{\tan\theta}\,d\theta$$

$\sin 2\theta = 2\sin\theta\cos\theta$

$$= \int \frac{1}{\sec^2\theta}\,d\theta$$

$x^2 - a^2$

$\rightsquigarrow x = a\sec\theta$

$$= \int \cos^2\theta\,d\theta$$

$\sec\theta = \frac{1}{\cos\theta}$

$$= \int \frac{1}{2} + \frac{1}{2}\cos(2\theta)\,d\theta$$

$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$

$$= \frac{1}{2}\theta + \frac{1}{4}\sin(2\theta) + C$$

$\leftarrow$ convert back to $x$'s

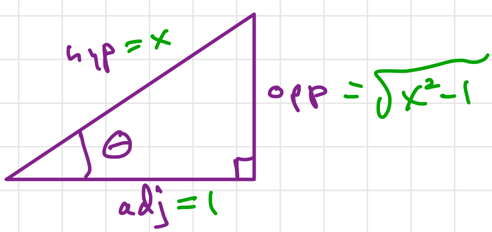
$$= \frac{1}{2}\theta + \frac{1}{4}\,2\sin\theta\cos\theta$$

$\theta = \sec^{-1}(x)$

$$= \frac{1}{2}\sec^{-1}(x) + \frac{1}{2}\,\frac{\sqrt{x^2-1}}{x}\,\frac{1}{x} + C$$

$$= \frac{1}{2}\sec^{-1}(x) + \frac{1}{2}\,\frac{\sqrt{x^2-1}}{x^2} + C$$

$\frac{x}{1} = \sec\theta = \frac{hyp}{adj}$




$\Rightarrow$
$$\begin{cases} \sin\theta = \frac{opp}{hyp} = \frac{\sqrt{x^2-1}}{x} \\ \cos\theta = \frac{1}{x} \end{cases}$$

example

$$\int \frac{1}{x^2\sqrt{x^2-1}}\,dx$$

$$\begin{cases} x = \sec\theta \\ x^2-1 = \sec^2\theta - 1 = \tan^2\theta \\ dx = \sec\theta\tan\theta\, d\theta \end{cases}$$

$$= \int \frac{1}{\sec^{\cancel{2}}\theta\ \cancel{\tan}\theta}\ \cancel{\sec}\theta\ \cancel{\tan}\theta\ d\theta$$

$$= \int \cos\theta\ d\theta$$

example $\displaystyle\int \frac{1}{x^4\sqrt{x^2-1}}\,dx =$

$$\int \frac{1}{\sec^4 x\tan x}\sec x\tan x\, dx = \int \cos^3(x)\,dx$$

$$= ??$$

$$= \int \cos^2 x\cos x\, dx$$

$$= \int (1-\sin^2 x)\cos x\, dx$$

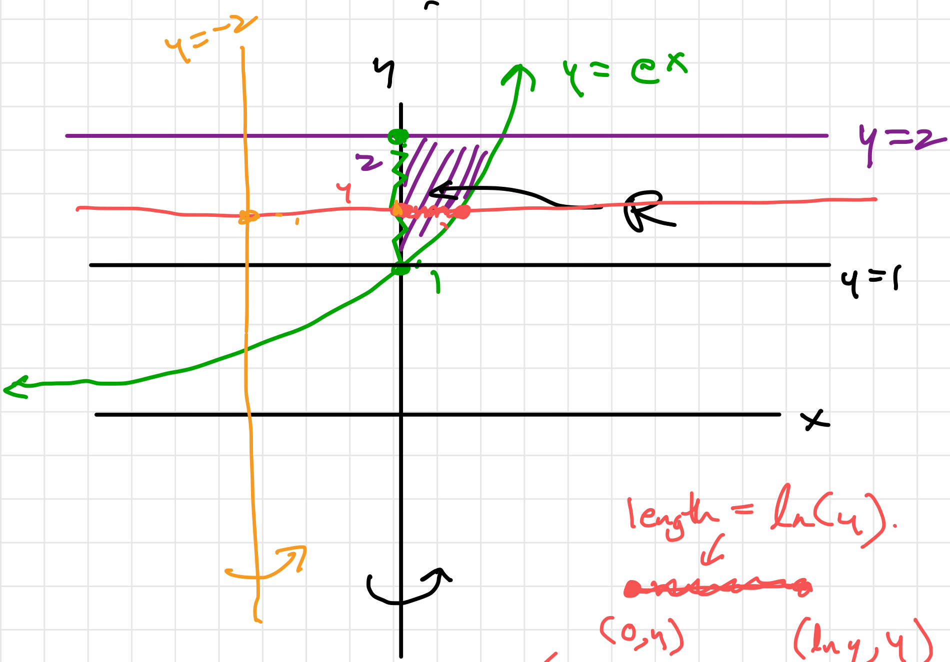
$$\left|\begin{array}{l} u = \sin x \\ du = \cos x\, dx \end{array}\right.$$

$$= \int 1-u^2\, du$$

$$= u - \frac{u^3}{3} + C = \sin x - \frac{\sin^3 x}{3} + C$$

$$= \frac{\sqrt{x^2-1}}{x} - \frac{(x^2-1)^{3/2}}{3x^3} + C$$

example




$y = e^x$

$x = \ln y$

length $= \ln(y)$.

(0, y) (ln y, y)

## disk method

axis of rotation = reference line
y-axis

Type II:

$$R: \begin{cases} 1 \le y \le 2 \\ 0 \le x \le \ln y \end{cases}$$

Volume

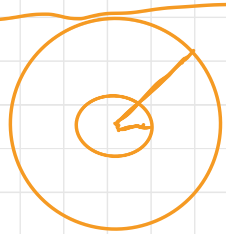
$$= \int_1^2 \pi (\ln y)^2 \, dy$$

rotate around y-axis

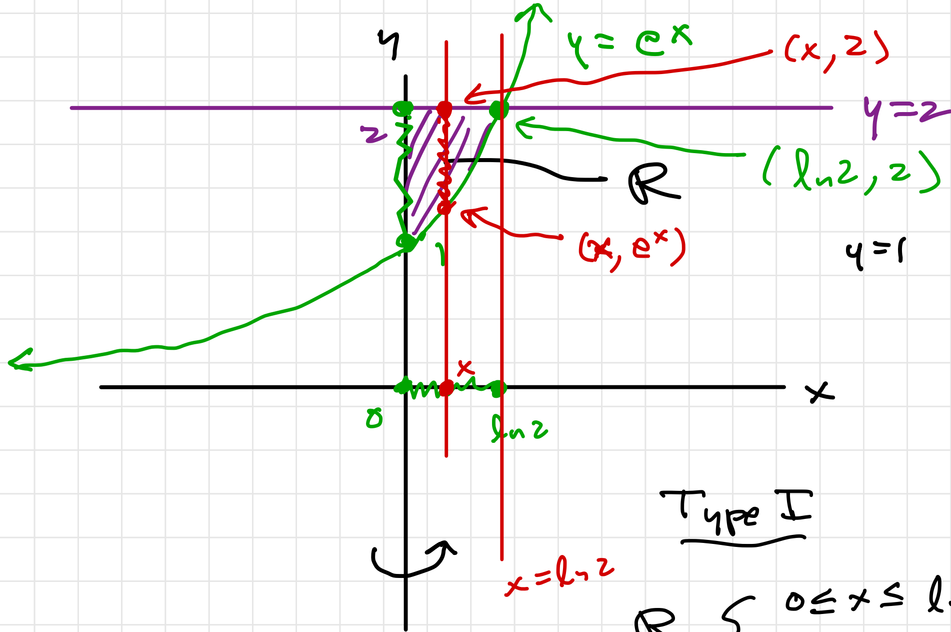
ln y

area(disk) $= \pi (\ln y)^2$

rotate around y = -2



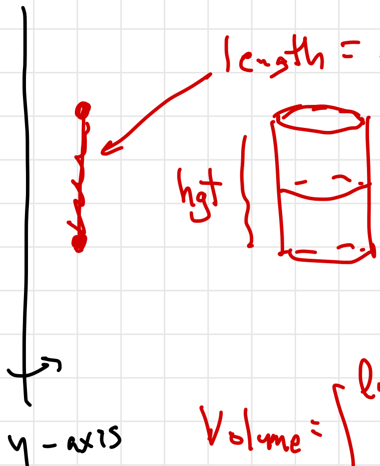
washer method




Type I

$$R: \begin{cases} 0 \le x \le \ln 2 \\ e^x \le y \le 2 \end{cases}$$

shell method reference line = x-axis




$2+x$

cylinder has radius $= x$ and height $= 2 - e^x$

area cylinder $= 2\pi x(2-e^x)$

$$\text{Volume} = \int_0^{\ln 2} 2\pi x(2-e^x)\, dx$$

IBP

# Next Try a few of these... →

## Even more problems in Section 7.5:

**49.** $\int \frac{1}{x\sqrt{4x+1}}\,dx$

**50.** $\int \frac{1}{x^2\sqrt{4x+1}}\,dx$

**51.** $\int \frac{1}{x\sqrt{4x^2+1}}\,dx$

**52.** $\int \frac{dx}{x(x^4+1)}$

**53.** $\int x^2 \sinh mx\,dx$

**54.** $\int (x+\sin x)^2\,dx$

**55.** $\int \frac{dx}{x+x\sqrt{x}}$

**56.** $\int \frac{dx}{\sqrt{x}+x\sqrt{x}}$

**57.** $\int x\sqrt[3]{x+c}\,dx$

**58.** $\int \frac{x\ln x}{\sqrt{x^2-1}}\,dx$

**59.** $\int \frac{dx}{x^4-16}$

**60.** $\int \frac{dx}{x^2\sqrt{4x^2-1}}$

**61.** $\int \frac{d\theta}{1+\cos\theta}$

**62.** $\int \frac{d\theta}{1+\cos^2\theta}$

**63.** $\int \sqrt{x}\,e^{\sqrt{x}}\,dx$

**64.** $\int \frac{1}{\sqrt{\sqrt{x}+1}}\,dx$

**65.** $\int \frac{\sin 2x}{1+\cos^4 x}\,dx$

**66.** $\int_{\pi/4}^{\pi/3} \frac{\ln(\tan x)}{\sin x\cos x}\,dx$

**67.** $\int \frac{1}{\sqrt{x+1}+\sqrt{x}}\,dx$

**68.** $\int \frac{x^2}{x^6+3x^3+2}\,dx$

**69.** $\int_1^{\sqrt{3}} \frac{\sqrt{1+x^2}}{x^2}\,dx$

**70.** $\int \frac{1}{1+2e^x-e^{-x}}\,dx$

**71.** $\int \frac{e^{2x}}{1+e^x}\,dx$

**72.** $\int \frac{\ln(x+1)}{x^2}\,dx$

**73.** $\int \frac{x+\arcsin x}{\sqrt{1-x^2}}\,dx$

**74.** $\int \frac{4^x+10^x}{2^x}\,dx$

**75.** $\int \frac{dx}{x\ln x - x}$

**76.** $\int \frac{x^2}{\sqrt{x^2+1}}\,dx$

**77.** $\int \frac{xe^x}{\sqrt{1+e^x}}\,dx$

**78.** $\int \frac{1+\sin x}{1-\sin x}\,dx$

**79.** $\int x\sin^2 x\cos x\,dx$

**80.** $\int \frac{\sec x\cos 2x}{\sin x+\sec x}\,dx$

**81.** $\int \sqrt{1-\sin x}\,dx$

**82.** $\int \frac{\sin x\cos x}{\sin^4 x+\cos^4 x}\,dx$

## A few hints:

#49 (see next page)

#50 $u^2 = 4x+1$

#51 trig substitution $x = \frac{1}{2}\tan\theta$

#52 $u = x^2$ observe that $\frac{1}{x(x^4+1)} = \frac{x}{x^2((x^2)^2+1)}$

#54 expand out

#55 $x = u^2$, $dx = 2u\,du$

#56 $x = u^2$

#57 $x + C = u^3$, $dx = 3u^2\,du$

#69 $u^2 = 1+x^2$ is easier than trig substitution

**55.** $\int \frac{dx}{x + x\sqrt{x}}$

rationalizing substitution

$$\begin{cases} \sqrt{x} = u \\ u^2 = x \\ dx = 2u\,du \end{cases}$$

$$= \int \frac{2u\,du}{u^2 + u^3}$$

$$= \int \frac{2u}{u^2(1+u)}\,du$$

$$= \int \frac{2}{u(1+u)}\,du$$

use partial fractions

$$\frac{2+0u}{u(1+u)} = \frac{A}{u} + \frac{B}{1+u}$$

$$= \frac{A(1+u) + Bu}{u(1+u)}$$

$$= \frac{A + u(A+B)}{u(1+u)}$$

$$\begin{cases} A = 2 \\ A + B = 0 \end{cases}$$

$$B = -2$$

$$= \int \frac{2}{u} - \frac{2}{1+u}\,du$$

$$= 2\ln|u| - 2\ln|1+u| + C$$

$$= 2\ln|\sqrt{x}| - 2\ln|1+\sqrt{x}| + C$$

**75.** $\int \frac{dx}{x \ln x - x}$ $= \int \frac{dx}{(\ln x - 1)x}$

$$u = \ln(x)$$

$$du = \frac{1}{x}\,dx$$

$$= \int \frac{1}{\ln(x) - 1} \frac{1}{x}\,dx$$

$$= \int \frac{1}{u-1}\,du$$

$$= \ln|u-1| + C = \ln|\ln(x) - 1| + C$$

**65.** $\displaystyle\int \frac{\sin 2x}{1+\cos^4 x}\,dx$

$\sin(2x) = 2\sin x \cos x$

$$= \int \frac{2\sin(x)\cos(x)}{1+\cos^4(x)}\,dx$$

$$\begin{cases} u = \cos x \\ du = -\sin x\,dx \end{cases}$$

$$= -\int \frac{2\cos(x)}{1+\cos^4(x)}(-\sin x)\,dx$$

$$= -\int \frac{2u}{1+u^4}\,du$$

$$\begin{cases} v = u^2 \\ dv = 2u\,du \end{cases}$$

$u^4 = v^2$

$$= -\int \frac{dv}{1+v^2}$$

$$= -\arctan(v) + C$$

$$= -\arctan(u^2) + C$$

$$= -\arctan(\cos^2 x) + C$$

**74.** $\displaystyle\int \frac{4^x + 10^x}{2^x}\,dx$

$$4^x = (2^2)^x = 2^{2x} = 2^x \cdot 2^x$$

$$10^x = (2 \cdot 5)^x = 2^x \cdot 5^x$$

$$\frac{4^x + 10^x}{2^x} = \frac{2^x \cdot \cancel{2^x} + \cancel{2^x} \cdot 5^x}{\cancel{2^x}} = 2^x + 5^x$$

$$\int 2^x\,dx = \frac{1}{\ln 2}\int e^{\ln(2)x}\,\ln 2\,dx \qquad \begin{cases} u = \ln(2)x \\ du = \ln(2)\,dx \end{cases}$$

$$= \int \frac{e^u}{\ln 2}\,du = \frac{1}{\ln 2} e^u + C$$

$$= \frac{1}{\ln 2} e^{\ln(2)x} + C = \frac{1}{\ln(2)} 2^x + C$$

$$\boxed{2^x = e^{x\ln(2)}}$$