White Exam

EXAM 2 **Name:**
Math 2433
10/13/21

Problem 1. *(25 points)* This problem involves the sequence $\{a_n\}_{n=1}^{\infty} = \left\{\dfrac{n^2+2}{n^2+1}\right\}_{n=1}^{\infty}$.

(a) Write out the first five terms of this sequence.

(b) Does the sequence $\{a_n\}_{n=1}^{\infty}$ converge or diverge. If it converges then find its limit as $n \to \infty$.

(c) Write out the first three partial sums of the series $\sum_{n=1}^{\infty} a_n$.

(d) What if anything does the Test for Divergence tell us about whether $\sum_{n=1}^{\infty} a_n$ converges or diverges? In your explanation include a clear statement of what this Test says.

(a) $\{a_n\}_{n=1}^{\infty} = \left\{ \frac{3}{2}, \frac{6}{5}, \frac{11}{10}, \frac{18}{17}, \frac{27}{26}, \dots \right\}$

(b) It converges. $\lim_{n\to\infty} \frac{n^2+2}{n^2+1} = \lim_{n\to\infty} \frac{1+2/n^2}{1+1/n^2} = \frac{1}{1} = 1$

(c) $\{s_n\}_{n=1}^{\infty} = \left\{ \frac{3}{2}, \frac{3}{2}+\frac{6}{5}, \frac{3}{2}+\frac{6}{5}+\frac{11}{10}, \dots \right\} = \left\{ \frac{3}{2}, \frac{27}{10}, \frac{19}{5}, \dots \right\}$

(d) $\sum_{n=1}^{\infty} \frac{n^2+2}{n^2+1}$ diverges because $\lim_{n\to\infty} \frac{n^2+2}{n^2+1} \neq 0$, using

Test for Divergence: If $\lim_{n\to\infty} a_n \neq 0$ then $\sum_{n=1}^{\infty} a_n$ diverges.

Problem 2. *(30 points)* An object moves along a curve C in the xy-plane so that at time t it is located at the point (x, y) where $x = t^4 - 2t^2$, $y = t^3 - 3t^2 - 9t$.

(a) Does the curve C pass through the origin? Explain.

(b) How many times (if any) does the object pass through the origin? Explain.

(c) For what values of t is the object moving to the left?

(d) For what values of t is the object moving downward?

(e) Determine the t-values for all points on C that have a horizontal tangent line.

(a) Yes. When $t=0$, $(x,y) = (0,0)$

(b) It only passes thru $(0,0)$ once (when $t=0$):

Suppose $(t^4-2t^2, t^3-3t^2-9t) = (0,0)$ then $t^4-2t^2 = 0 \Rightarrow t^2(t^2-2)=0$

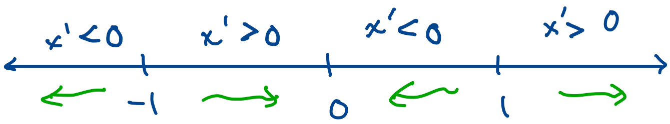
So $t = 0, -\sqrt{2}$, or $\sqrt{2}$.

$t=-\sqrt{2} \Rightarrow t^3-3t^2-9t = -2\sqrt{2} - 6 + 9\sqrt{2} = 7\sqrt{2}-6 \neq 0$.

$t=\sqrt{2} \Rightarrow t^3-3t^2-9t = -7\sqrt{2}-6 \neq 0$

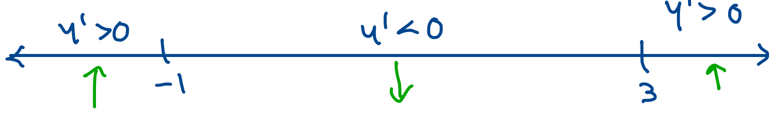
This shows that the only value of t with $(t^4-2t^2, t^3-3t^2-9t) = (0,0)$ is $t=0$.

(c) $\frac{dx}{dt} = 4t^3 - 4t = 4t(t-1)(t+1)$. So $x(t)$ has critical #'s $0, 1, -1$.



Object moving left when $t \leq -1$ or $0 \leq t \leq 1$.

(d) $\frac{dy}{dt} = 3t^2 - 6t - 9 = 3(t-3)(t+1)$. So $y(t)$ has critical #'s $-1, 3$



Object moving down when $-1 \leq t \leq 3$.

(e) The curve C will have a horizontal tangent for any t-value where $dy/dx = 0$. Here

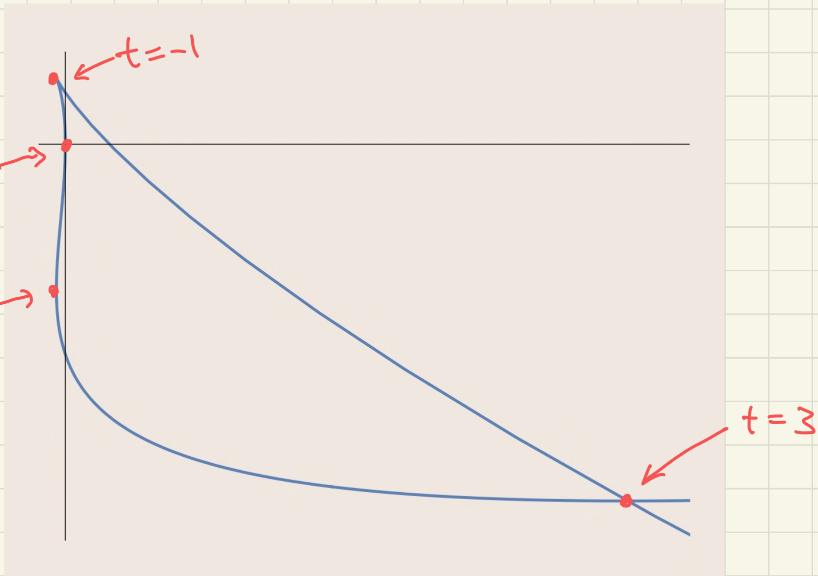
$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3(t-3)(t+1)}{4t(t-1)(t+1)} = \frac{3(t-3)}{4t(t-1)}$$

and we can see that $dy/dx = 0$ only when $t = 3$.

So C has a horizontal tangent only when $t=3$,

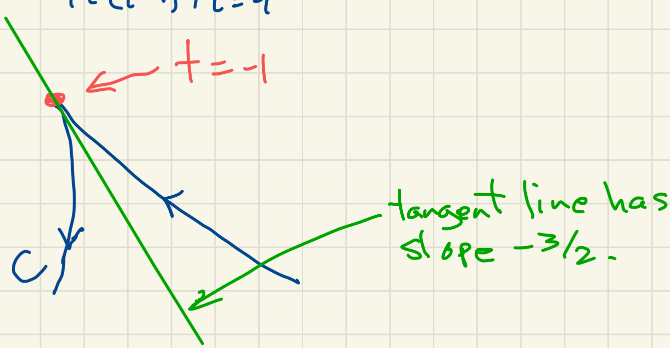
But this is a little tricky! See next page to understand more...

Since $dy/dt = 3(t-3)(t+1)$ equals 0 when $t=-1$ we might initially guess that there is also a horizontal tangent when $t=-1$. However dx/dt also equals 0 when $t=-1$. This means that the object has stopped when $t=-1$ (and the curve C has a "cusp" there). Here's a picture:



Notice that there actually is a tangent line to C at the cusp. It has slope

$$\frac{dy}{dx}\bigg|_{t=-1} = \frac{3(t-3)}{4t(t-1)}\bigg|_{t=-1} = \frac{-12}{8} = -3/2$$



PROBLEM 3. *(25 points)* A curve C in the xy-plane has polar coordinate equation $r = 4 + 2\cos(\theta)$. Let P be the point on C with $\theta = \pi/2$.

(a) Give the polar coordinates (r, θ) and the rectangular coordinates (x, y) for P.

(b) Find the slope of the line tangent to C at P.

(c) Give the slope-intercept form equation for the line tangent to C at P.

(a) $(r,\theta) = (4, \pi/2)$ and $(x,y) = (4\cos\frac{\pi}{2}, 4\sin\frac{\pi}{2}) = (0,4)$

(b) $\begin{cases} x = (4+2\cos\theta)\cos\theta \\ y = (4+2\cos\theta)\sin\theta \end{cases}$

* use the product rule

$dx/d\theta \overset{*}{=} -2\sin\theta\cos\theta - (4+2\cos\theta)\sin\theta = -4\sin\theta - 4\sin\theta\cos\theta$

$dy/d\theta \overset{*}{=} -2\sin\theta\sin\theta + (4+2\cos\theta)\cos\theta = 4\cos\theta - 2\sin^2\theta + 2\cos^2\theta$

$$dy/dx = \frac{4\cos\theta - 2\sin^2\theta + 2\cos^2\theta}{-4\sin\theta - 4\sin\theta\cos\theta}$$

slope of tangent line at P $= \left.\frac{dy}{dx}\right|_{\theta=\pi/2} = \frac{4\cdot 0 - 2\cdot 1^2 + 2\cdot 0^2}{-4\cdot 1 - 4\cdot 1\cdot 0} = \frac{1}{2}$

(c) $y - 4 = \frac{1}{2}(x - 0)$

$\Rightarrow \quad y = \frac{1}{2}x + 4$

PROBLEM 4. *(25 points)* Determine whether each of the following geometric series converges or diverges, and find its sum if it does converge.

(a) $\sum_{n=1}^{\infty} \frac{8^{n+1}}{3^{n+1}}$ Diverges. (Geometric Series with $r = 8/3 > 1$)

(b) $\sum_{n=1}^{\infty} \frac{3^{n+1}}{8^{n+1}}$ Converges. (Geometric Series with $r = 3/8$, $-1 < r < 1$)

$$= \left(\frac{3}{8}\right)^2 + \left(\frac{3}{8}\right)^3 + \left(\frac{3}{8}\right)^4 + \cdots \quad = \left(1 + \left(\frac{3}{8}\right) + \left(\frac{3}{8}\right)^2 + \left(\frac{3}{8}\right)^3 + \left(\frac{3}{8}\right)^4 + \cdots\right) - \left(1 + \frac{3}{8}\right)$$

$$= \frac{1}{1-\frac{3}{8}} - 1 - \frac{3}{8} = 9/40$$

(c) $\sum_{n=3}^{\infty} \frac{8^{n-2}}{3^{n-2}}$ Diverges (Geometric Series with $r = 8/3$.)

(d) $\sum_{n=3}^{\infty} \frac{8^{n+1}}{3^{n+1}}$ Diverges (Geometric Series with $r = 8/3$.)

(e) $\sum_{n=1}^{\infty} (-1)^n \frac{3^{n+2}}{8^{n+1}}$ Converges. (Geometric Series with $r = -\frac{3}{8}$, $-1 < r < 1$)

$$= \sum_{n=1}^{\infty} \frac{9}{8}\left(-\frac{3}{8}\right)^n = \frac{9}{8}\left(\left(-\frac{3}{8}\right) + \left(-\frac{3}{8}\right)^2 + \left(-\frac{3}{8}\right)^3 + \cdots\right)$$

$$= \frac{9}{8}\left(1 + \left(-\frac{3}{8}\right) + \left(-\frac{3}{8}\right)^2 + \left(-\frac{3}{8}\right)^3 + \cdots\right) - \frac{9}{8}\cdot 1$$

$$= \frac{9}{8}\left(\frac{1}{1-(-3/8)}\right) - \frac{9}{8} = -\frac{27}{88}$$