Yellow Exam

EXAM 2 **Name:**
Math 2433
10/13/21

PROBLEM 1. *(25 points)* This problem involves the sequence $\{a_n\}_{n=1}^{\infty} = \left\{\frac{n^2-1}{n^2+2}\right\}_{n=1}^{\infty}$.

(a) Write out the first five terms of this sequence.

(b) Does the sequence $\{a_n\}_{n=1}^{\infty}$ converge or diverge. If it converges then find its limit as $n \to \infty$.

(c) Write out the first three partial sums of the series $\sum_{n=1}^{\infty} a_n$.

(d) What if anything does the Test for Divergence tell us about whether $\sum_{n=1}^{\infty} a_n$ converges or diverges? In your explanation include a clear statement of what this Test says.

(a) $\{a_n\}_{n=1}^{\infty} = \{0, \frac{3}{6}, \frac{8}{11}, \frac{15}{18}, \frac{24}{27}, \dots\} = \{0, \frac{1}{2}, \frac{8}{11}, \frac{5}{6}, \frac{8}{9}, \dots\}$

(b) It converges. $\lim_{n\to\infty} \frac{n^2-1}{n^2+2} = \lim_{n\to\infty} \frac{1-1/n^2}{1+2/n^2} = \frac{1}{1} = 1$

(c) $\{s_n\}_{n=1}^{\infty} = \{0, 0+\frac{1}{2}, 0+\frac{1}{2}+\frac{8}{11}, \dots\} = \{0, 1/2, \frac{27}{22}, \dots\}$

(d) $\sum_{n=1}^{\infty} \frac{n^2-1}{n^2+2}$ diverges because $\lim_{n\to\infty} \frac{n^2-1}{n^2+2} \neq 0$, using

Test for Divergence: If $\lim_{n\to\infty} a_n \neq 0$ then $\sum_{n=1}^{\infty} a_n$ diverges.

PROBLEM 2. *(30 points)* An object moves along a curve C in the xy-plane so that at time t it is located at the point (x, y) where $x = 2t^2 - t^4$, $y = t^3 - 3t^2 - 9t$.

(a) Does the curve C pass through the origin? Explain.

(b) How many times (if any) does the object pass through the origin? Explain.

(c) For what values of t is the object moving to the left?

(d) For what values of t is the object moving downward?

(e) Determine the t-values for all points on C that have a horizontal tangent line.

(a) Yes. When $t=0$, $(x,y) = (2\cdot 0^2 - 0^4,\ 0^3 - 3\cdot 0^2 - 9\cdot 0) = (0,0)$

(b) It only passes thru $(0,0)$ once (when $t=0$):

Suppose $(2t^2-t^4,\ t^3-3t^2-9t) = (0,0)$ then $2t^2-t^4=0 \Rightarrow t^2(2-t^2)=0$

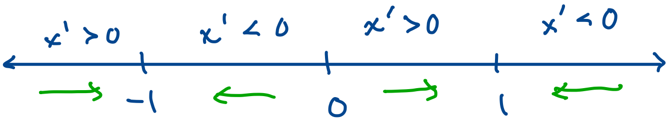
So $t = 0, -\sqrt{2}$, or $\sqrt{2}$.

$t=-\sqrt{2} \Rightarrow t^3-3t^2-9t = -2\sqrt{2} - 6 + 9\sqrt{2} = 7\sqrt{2}-6 \neq 0$.

$t=\sqrt{2} \Rightarrow t^3-3t^2-9t = -7\sqrt{2}-6 \neq 0$

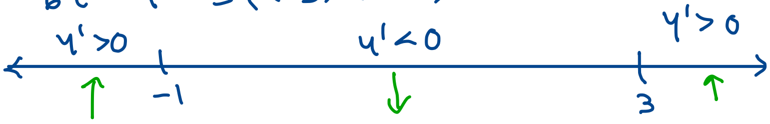
This shows that the only value of t with $(2t^2-t^4,\ t^3-3t^2-9t) = (0,0)$ is $t=0$.

(c) $\dfrac{dx}{dt} = 4t - 4t^3 = -4t(t-1)(t+1)$. So $x(t)$ has critical #'s $0, 1, -1$.



Object moving left when $-1 \le t \le 0$ or $t \ge 1$.

(d) $\dfrac{dy}{dt} = 3t^2 - 6t - 9 = 3(t-3)(t+1)$. So $y(t)$ has critical #'s $-1, 3$.



Object moving down when $-1 \le t \le 3$.

(e) The curve C will have a horizontal tangent for any t-value where $dy/dx = 0$. Here

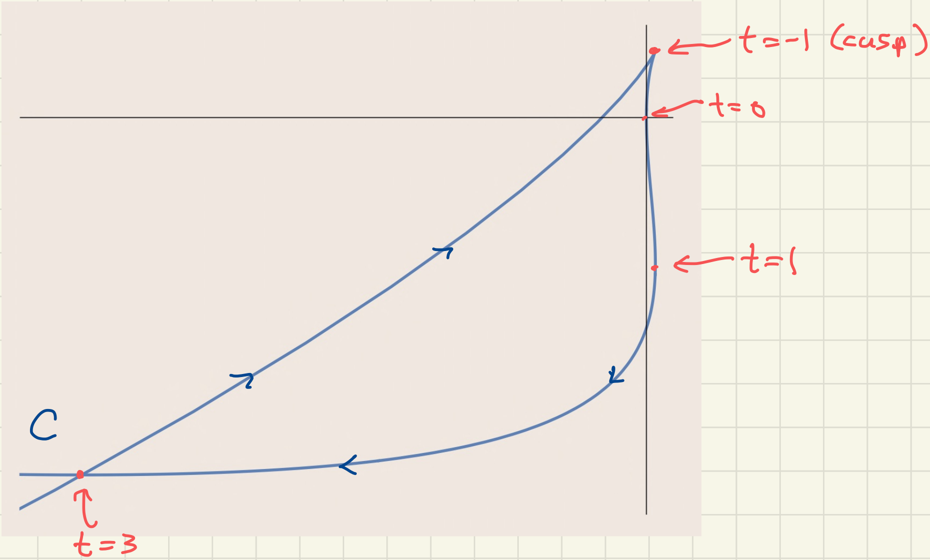
$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3(t-3)(t+1)}{-4t(t-1)(t+1)} = \frac{3(t-3)}{-4t(t-1)}$$

and we can see that $dy/dx = 0$ only when $t = 3$.

So C has a horizontal tangent only when $t=3$,

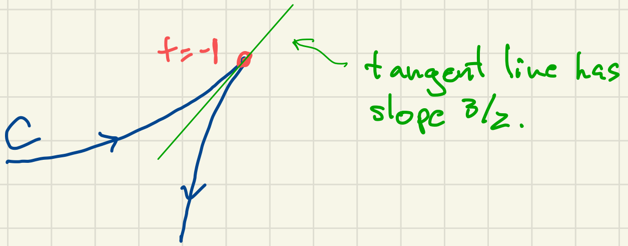
But this is a little tricky!! See next page if you want to see what is really happening.

Since $dy/dt = 3(t-3)(t+1)$ equals 0 when $t=-1$ we might initially guess that there is also a horizontal tangent when $t=-1$. However dx/dt also equals 0 when $t=-1$. This means that the object has stopped when $t=-1$ (and the curve C has a "cusp" there). Here's a picture:



Notice that there actually is a tangent line to C at the cusp. It has slope

$$\left.\frac{dy}{dx}\right|_{t=-1} = \left.\frac{3(t-3)}{-4t(t-1)}\right|_{t=-1} = \frac{-12}{-8} = \frac{3}{2}$$



PROBLEM 3. *(25 points)* A curve C in the xy-plane has polar coordinate equation $r = 6 + 3\cos(\theta)$. Let P be the point on C with $\theta = \pi/2$.

(a) Give the polar coordinates (r, θ) and the rectangular coordinates (x, y) for P.

(b) Find the slope of the line tangent to C at P.

(c) Give the slope-intercept form equation for the line tangent to C at P.

(a) $(r, \theta) = (6, \pi/2)$ and $(x, y) = (0, 6)$

* use the product rule

(b)
$$\begin{cases} x = (6 + 3\cos\theta)\cos\theta \\ y = (6 + 3\cos\theta)\sin\theta \end{cases}$$

$$dx/d\theta \overset{*}{=} -3\sin\theta\cos\theta - (6 + 3\cos\theta)\sin\theta = -6\sin\theta - 6\sin\theta\cos\theta$$

$$dy/d\theta \overset{*}{=} -3\sin\theta\sin\theta + (6 + 3\cos\theta)\cos\theta = 6\cos\theta - 3\sin^2\theta + 3\cos^2\theta$$

$$dy/dx = \frac{6\cos\theta - 3\sin^2\theta + 3\cos^2\theta}{-6\sin\theta - 6\sin\theta\cos\theta}$$

slope of tangent line at P $= \left.\dfrac{dy}{dx}\right|_{\theta = \pi/2} = \dfrac{0 - 3\cdot 1 + 0}{-6\cdot 1 - 6\cdot 1\cdot 0} = \dfrac{-3}{-6} = \dfrac{1}{2}$

(c) $y - 6 = \frac{1}{2}(x - 0)$

$\Rightarrow \quad y = \frac{1}{2}x + 6$

PROBLEM 4. *(25 points)* Determine whether each of the following geometric series converges or diverges, and find its sum if it does converge.

(a) $\sum_{n=1}^{\infty} \frac{9^{n+1}}{4^{n+1}}$ Diverges. (Geometric Series with $r = 9/4 > 1$)

(b) $\sum_{n=1}^{\infty} \frac{4^{n+1}}{9^{n+1}}$ Converges. (Geometric Series with $r = 4/9$, $-1<r<1$)

$= \left(\frac{4}{9}\right)^2 + \left(\frac{4}{9}\right)^3 + \left(\frac{4}{9}\right)^4 + \cdots = \left(1 + \left(\frac{4}{9}\right) + \left(\frac{4}{9}\right)^2 + \left(\frac{4}{9}\right)^3 + \left(\frac{4}{9}\right)^4 + \cdots\right) - \left(1 + \frac{4}{9}\right)$

$= \frac{1}{1-\frac{4}{9}} - 1 - \frac{4}{9} = \frac{16}{45}$

(c) $\sum_{n=3}^{\infty} \frac{9^{n-2}}{4^{n-2}}$ Diverges (Geometric Series with $r = 9/4$.)

(d) $\sum_{n=3}^{\infty} \frac{9^{n+1}}{4^{n+1}}$ Diverges (Geometric Series with $r = 9/4$.)

(e) $\sum_{n=1}^{\infty} (-1)^n \frac{4^{n+2}}{9^{n+1}}$ Converges. (Geometric Series with $r = -4/9$.)

$= \sum_{n=1}^{\infty} \frac{16}{9} \left(-\frac{4}{9}\right)^n = \frac{16}{9}\left(-\frac{4}{9} + \left(-\frac{4}{9}\right)^2 + \left(-\frac{4}{9}\right)^3 + \cdots \right)$

$= \frac{16}{9}\left(1 + \left(-\frac{4}{9}\right) + \left(-\frac{4}{9}\right)^2 + \left(-\frac{4}{9}\right)^3 + \cdots \right) - \frac{16}{9} \cdot 1$

$= \frac{16}{9}\left(\frac{1}{1-(-4/9)} \right) - 16/9 = -\frac{64}{117}$