EXAM 4 **Name:**
Math 2433
11/22/21

Problem 1. *(25 points)* Let ℓ be the line in 3-space containing the points $P = (-2, 0, 3)$ and $Q = (-1, 5, 2)$.
(a) Determine a scalar parametrization for ℓ.
(b) Determine a vector parametrization for ℓ.
(c) Does ℓ go through the point $R = (1, 2, 3)$? Explain.

(a) A direction vector for ℓ is $\vec{d} = \overrightarrow{PQ} = \langle 1, 5, -1 \rangle$.
Using this and P gives

$$\ell = \begin{cases} x = t-2 \\ y = 5t \\ z = -t+3 \end{cases}$$

as one possible parametrization.
(There are many other possible parametrizations.)

(b) $\ell : \vec{r}(t) = \langle t-2, 5t, -t+3 \rangle$
(same as $\vec{r}(t) = \langle 1, 5, -1 \rangle t + \langle -2, 0, 3 \rangle$

(c) ℓ does not go through $(1,2,3)$ because the equations

$$\begin{cases} t-2=0 \\ 5t=0 \\ -t+3=0 \end{cases}$$ have no solution for t.

Problem 2. *(5 points)* Explain why it is incorrect to refer "THE" direction vector for a line in 3-space rather than "A" direction vector. Give a brief example supporting your explanation.

If $\vec{d}$ is one direction vector for ℓ then so is any parallel vector $k\vec{d}$ where $k \neq 0$.

For example in the previous problem the vectors

$$\langle 1, 5, -1 \rangle, \langle -1, -5, 1 \rangle, \langle 4, 20, -4 \rangle, \langle e^2, 5e^2, -e^2 \rangle$$

are all direction vectors for the given line ℓ.

PROBLEM 3. *(25 points)* Let $\mathcal{P}$ be the plane containing the three non-collinear points

$$P = (-2, 3, 0), \quad Q = (1, -2, 2), \quad \text{and} \quad R = (2, 2, 2).$$

(a) Find a linear equation in three variables describing $\mathcal{P}$.
(b) Find two distinct points that are on the line of intersection of $\mathcal{P}$ with xz-coordinate plane.
(c) Is the plane $\mathcal{P}$ parallel to the plane $x + y + z = 17$? Do these two planes intersect in a line? Explain.

(a) $\overrightarrow{PQ} = \langle 3, -5, 2 \rangle$ and $\overrightarrow{PR} = \langle 4, -1, 2 \rangle$ are vectors in $\mathcal{P}$
so $\overrightarrow{PQ} \times \overrightarrow{PR} = \langle -8, 2, 17 \rangle$ is a normal vector for $\mathcal{P}$.

check $\langle 3, -5, 2 \rangle \cdot \langle -8, 2, 17 \rangle = -24 - 10 + 34 = 0$ and
$\langle 4, -1, 2 \rangle \cdot \langle -8, 2, 17 \rangle = 0$

An equation for the plane is

$$-8(x+2) + 2(y-3) + 17(z-0) = 0$$

$$-8x + 2y + 17z + (-16 - 6) = 0$$

or, $\mathcal{P}: -8x + 2y + 17z - 22 = 0$

check $(x,y,z) = (-2,3,0) \Rightarrow -8(-2) + 2(3) + 17(0) - 2 = 0$
and $-8(1) + 2(-2) + 17(2) - 22 = 0$, and $-8(2) + 2(2) + 17(2) - 22 = 0$

(b) Find solutions to $\begin{cases} y = 0 & \leftarrow xz\text{-plane} \\ -8x + 2y + 17z = 22 \end{cases} \Rightarrow -8x + 17z = 22$

For example, $(x, y, z) = (0, 0, \frac{22}{17})$ and $(-\frac{11}{4}, 0, 0)$ work.

(c) A normal vector for the plane $x + y + z = 17$ is $\langle 1, 1, 1 \rangle$
which is not parallel to $\vec{N}_{\mathcal{P}} = \langle -8, 2, 17 \rangle$, so the two planes are neither equal nor parallel.
Two planes that are neither equal nor parallel will intersect in a line.

Problem 4. *(25 points)* Let $\ell : x = 2t - 3, y = -3t + 1, z = t - 4$ be a scalar parametrization for a line ℓ and let ℓ' be the line parallel to ℓ which passes through the origin.
(a) Find a parametrization for ℓ'.
(b) Give an equation for the plane which contains both ℓ and ℓ'.
(c) Find the point of intersection of ℓ with the plane $x - y + z = 0$, if there is one.

(a) $$\ell' : \begin{cases} x = 2t \\ y = -3t \\ z = t \end{cases}$$

(b) $P = (-3, 1, -4)$ is on ℓ, and $O = (0,0,0)$ is on ℓ'
so $\vec{PQ} = \langle 3, -1, 4 \rangle$ and $\vec{d}_\ell = \langle 2, -3, 1 \rangle$ are vectors parallel to the plane p containing ℓ and ℓ'. So

$$\vec{N}_p = \langle 3, -1, 4 \rangle \times \langle 2, -3, 1 \rangle = \langle 11, 5, -7 \rangle$$

is a normal vector for p. Then

$$p : 11(x-0) + 5(y-0) - 7(z-0) = 0$$
$$\Rightarrow p : 11x + 5y - 7z = 0$$

check:
Plug in the points $(x,y,z) = (0,0,0)$ and $(x,y,z) = (2t-3, -3t+1, t-4)$
....

(c) Solving $x = 2t-3$, $y = -3t+1$, $z = t-4$, $x - y + z = 0$ for t gives
$$0 = (2t-3) - (-3t+1) + (t-4) = 6t - 8 \Rightarrow t = 4/3$$
Point of intersection is $(-\frac{1}{3}, -3, -\frac{8}{3})$.

check: $$(-\tfrac{1}{3}) - (-3) + (-\tfrac{8}{3}) = 0$$

Problem 5. *(25 points)* Let C be the curve describing the motion of an object in 3-space with vector function

$$\mathbf{r}(t) = \langle t^3 + 1, 3t^2 - 1, t^2 - 2t^3 \rangle$$

(a) Give an example of a point P which is on the curve C and a point Q which is not on the curve C.
(b) What the velocity function $\mathbf{r}'(t) = \mathbf{v}(t)$?
(c) Find a parametrization for the line ℓ which is tangent to C at the point you determined in part (a).
(d) For which values of t is the object moving upwards?
(e) Give a formula for the speed of the object at time t.
(f) Are there any times at which the object is at rest, and where is it located at those times?

(a) $P = (2, 2, -1)$ is the point on C where $t = 1$.
$O = (0,0,0)$ is not on C (because there is no value of t with $t^3 + 1 = 0$, $3t^2 - 1 = 0$, $t^2 - 2t^3 = 0$.)

(b) $\vec{r}\,'(t) = \langle 3t^2, 6t, 2t - 6t^2 \rangle$

(c) $\vec{r}\,'(1) = \langle 3, 6, -4 \rangle = \vec{d}_\ell$ and

ℓ: $x = 3t + 2$, $y = 6t + 2$, $z = -4t - 1$

(d) The object is moving upwards when

$$z'(t) = 2t - 6t^2 = 2t(1 - 3t) > 0$$

and this happens when $0 \le t \le 1/3$.

$\sqrt{t^2} = |t|$

(e) $\text{speed}(t) = |\vec{r}\,'(t)| = \left((3t^2)^2 + (6t)^2 + (2t - 6t^2)^2 \right)^{1/2}$

$= (45t^4 - 24t^3 + 40t^2)^{1/2} = |t| \, (45t^2 - 24t + 40)^{1/2}$

(f) $\text{speed}(t) = 0$ if either $t = 0$ or $45t^2 - 24t + 40 = 0$.
But the quadratic $45t^2 - 24t + 40$ never equals 0 because its discriminant is $24^2 - 4(45)(40)$ which is negative. So object at rest only when $t = 0$. The point is $(x(0), y(0), z(0)) = (1, -1, 0)$

See further discussion below.

PROBLEM 5. *(25 points)* Let C be the curve describing the motion of an object in 3-space with vector function

$$\mathbf{r}(t) = \langle t^3 + 1, 3t^2 - 1, t^2 - 2t^3 \rangle$$

(a) Give an example of a point P which is on the curve C and a point Q which is not on the curve C.
(b) What the velocity function $\mathbf{r}'(t) = \mathbf{v}(t)$?
(c) Find a parametrization for the line ℓ which is tangent to C at the point you determined in part (a).
(d) For which values of t is the object moving upwards?
(e) Give a formula for the speed of the object at time t.
(f) Are there any times at which the object is at rest, and where is it located at those times?

If, in part (a), we had made the unfortunate choice of the point $P = (1, -1, 0)$ where $t = 0$, then we would have discovered that $\vec{r}\,'(0) = \vec{0}$ which is <u>not</u> an allowable direction vector for ℓ (direction vectors must be non-zero).

From part (f) we can see that the object has come to a stop when $t = 0$, and the curve C has a "cusp" at $P = (1, -1, 0)$. Nevertheless there is a tangent line to C at P and it has equations

$$\ell: \ x = 1, \ y = 6t - 1, \ z = 2t$$

<u>Explanation</u> The vector $\vec{r}\,'(t) = t\langle 3t, 6, 2-6t \rangle$, which is tangent to C at the point with time t, is parallel with the vector $\langle 3t, 6, 2-6t \rangle$.

Taking $t = 0$ shows that $\langle 0, 6, 2 \rangle$ is a direction vector for the line tangent to C at $P = (1, -1, 0)$.

schematic picture of cusp