Infinite Series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + \cdots$

Quick Recap:

There are three important objects associated with a series:

I. The series itself $\sum_{n=1}^{\infty} a_n$.

II. The sequence of terms $\{a_n\}_{n=1}^{\infty}$.

III. The sequence of partial sums $\{s_n\}_{n=1}^{\infty}$.

(where $s_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{k=1}^{n} a_k$.)

$\boxed{\text{BQ}}$ BASIC QUESTION:

Does a given series $\sum_{n=1}^{\infty} a_n$ converge or diverge?

Some examples:

Converge: $\sum_{n=1}^{\infty} 0$, $\sum_{n=1}^{\infty} \left(\frac{1}{n+4} - \frac{1}{n+5}\right)$, $\sum_{n=1}^{\infty} \frac{1}{n^2}$, $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$

Diverge: $\sum_{n=1}^{\infty} 1$, $\sum_{n=1}^{\infty} (-1)^{n+1}$, $\sum_{n=1}^{\infty} \frac{1}{n}$

rephrase:

$\boxed{\text{BQ}}$ The Basic Question About Infinite Series:

How can we tell whether or not $\sum_{n=1}^{\infty} a_n$ converges by just looking at the sequence of terms $\{a_n\}_{n=1}^{\infty}$?

To start to address $\boxed{BQ}$ we have an important theorem and an important type of example (called geometric series)

Theorem (Test for Divergence)

If $\lim_{n\to\infty} a_n \neq 0$ then the series $\sum_{n=1}^{\infty} a_n$ diverges.

examples

① $\sum_{n=1}^{\infty} \frac{n}{10n+5}$ diverges b/c $\lim_{n\to\infty} \frac{n}{10n+5} = \lim_{n\to\infty} \frac{1}{10+5/n} = \frac{1}{10}$ and $\frac{1}{10} \neq 0$.

Logic behind this (and behind the Test for divergence):
The partial sum $s_n = \sum_{k=1}^{n} \frac{n}{10n+5}$ is adding together n numbers each of which is approximately equal to $\frac{1}{10}$. So $s_n \approx \frac{1}{10} n$ and $\frac{1}{10} n \to \infty$ as $n \to \infty$.

② $\sum_{n=1}^{\infty} (-1)^n$ diverges because $\lim_{n\to\infty} (-1)^n = \text{DNE}$.

Comment: The Test for Divergence result is important and it should be considered first in addressing $\boxed{BQ}$. But it is totally inconclusive for series $\sum_{n=1}^{\infty} a_n$ where $\lim_{n\to\infty} a_n = 0$.

③ $\sum_{n=1}^{\infty} \frac{1}{n}$. In this example $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$ but we will show later that this series diverges.

Comment Sections 11.3 – 11.7 are devoted to answering $\boxed{BQ}$ for series $\sum_{n=1}^{\infty} a_n$ where $\lim_{n\to\infty} a_n = 0$. The Test for Divergence tells us <u>nothing</u> about these series.

Geometric Series $\quad \sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots$

(i) When $|r|<1$ the geometric series converges and

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}.$$

(ii) When $|r| \geq 1$, $\sum_{n=1}^{\infty} ar^{n-1}$ diverges.

Explanation for (i): $\sum_{n=1}^{\infty} ar^{n-1}$ has n^{th} partial sum

$$\begin{aligned} S_n &= a + ar + ar^2 + \cdots + ar^{n-1} \\ &= a(1+r+r^2+\cdots+r^{n-1}) \quad \leftarrow \text{Explain } * \\ &= a\frac{1-r^n}{1-r} \\ &= \frac{a}{1-r} - \frac{a}{1-r}r^n \xrightarrow{n\to\infty} \frac{a}{1-r} \end{aligned}$$

$*$ $(1-r)(1+r+r^2+\cdots+r^{n-1})$

$= (1+r+r^2+\cdots+r^{n-1}) - (r+r^2+r^3+\cdots+r^n)$

$= 1-r^n$

Examples

① $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = \frac{1}{1-1/2} = 2$

For $|r|<1$, Note that $\frac{a}{1-r} = \sum_{n=1}^{\infty} a r^{n-1} = a\sum_{m=0}^{\infty} r^m$.

I will often write the geometric series as

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \text{ for } |x|<1.$$

(with indexing starting at 0).

If this seems confusing, it often makes things clearer to write the sums out using ellipses (....). For example,

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots = \sum_{n=1}^{\infty} x^{n-1}.$$

examples (Explanations below.)

② $\sum_{n=2}^{\infty} (2/3)^n = 4/3$

③ $\sum_{n=2}^{\infty} (3/2)^n =$ diverges

④ $\sum_{n=1}^{\infty} \frac{5^{n+1}}{10^{2n-1}} = 50/19$

⑤ $\sum_{n=0}^{\infty} \frac{2^{2n}+3}{7^{n+2}} = \sum_{n=0}^{\infty} \frac{2^{2n}}{7^{n+2}} + \sum_{n=0}^{\infty} \frac{3}{7^{(n+2)}}$

$$= \frac{1}{21} + \frac{1}{14} = 5/42$$

(2) method #1:

$$\sum_{n=2}^{\infty}\left(2/3\right)^n = (2/3)^2+(2/3)^3+(2/3)^4+\cdots = \left(\frac{2}{3}\right)^2\left(1+\left(\frac{2}{3}\right)+\left(\frac{2}{3}\right)^2+\cdots\right)$$

$$= \frac{4}{9}\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^{n-1} = \frac{4}{9}\left(\frac{1}{1-2/3}\right) = 4/3$$

taking $r = 2/3$

(2) method #2.

$$\sum_{n=2}^{\infty}\left(2/3\right)^n = \sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n - \underbrace{\left(\frac{2}{3}\right)^0}_{n=0\text{ term}} - \underbrace{\left(\frac{2}{3}\right)^1}_{n=1\text{ term}} = \frac{1}{1-\frac{2}{3}} - 1 - \frac{2}{3} = 4/3$$

(3) $\sum_{n=2}^{\infty}(3/2)^n$ diverges because $r = 3/2 \geq 1$.

(4) $$\frac{5^{n+1}}{10^{2n-1}} = \frac{5}{10^{-1}}\frac{5^n}{10^{2n}} = 50\cdot\frac{5^n}{100^n} = 50\left(\frac{5}{100}\right)^n = 50\cdot\left(\frac{1}{20}\right)^n$$

So $$\sum_{n=1}^{\infty}\frac{5^{n+1}}{10^{2n-1}} = \sum_{n=1}^{\infty}50\left(\frac{1}{20}\right)^n = 50\left(\sum_{n=0}^{\infty}\left(\frac{1}{20}\right)^n - 1\right)$$

$r = \frac{1}{20}$

$$= 50\left(\frac{1}{1-\frac{1}{20}} - 1\right) = 50/19$$

(5) $$\sum_{n=0}^{\infty}\frac{2^{2n}}{7^{n+2}} = \frac{1}{49}\sum_{n=0}^{\infty}\left(\frac{4}{7}\right)^n = \frac{1}{49}\,\frac{1}{1-4/7} = \frac{1}{21}$$

$$\sum_{n=0}^{\infty}\frac{3}{7^{n+2}} = \frac{3}{49}\sum_{n=0}^{\infty}\left(\frac{1}{7}\right)^n = \frac{3}{49}\left(\frac{1}{1-1/7}\right) = \frac{1}{14}$$

Some of these examples have used these:

Convenient Facts:

If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent series. Then $\sum_{n=1}^{\infty} c a_n$ and $\sum_{n=1}^{\infty} (a_n + b_n)$ are convergent, and

(a) $\sum_{n=1}^{\infty} c a_n = c \sum_{n=1} a_n$

(b) $\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$