For an infinite series there are 3 associated objects:

① The series itself: $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$

② The sequence of terms: $\{a_n\}_{n=1}^{\infty} = \{a_1, a_2, a_3, \ldots\}$

③ The sequence of partial sums: $\{S_n\}_{n=1}^{\infty}$ where

$$S_n = a_1 + a_2 + \cdots + a_n$$

If $\{S_n\}$ has a finite limit as $n \to \infty$ then we say $\sum_{n=1}^{\infty} a_n$ converges, and $\lim_{n\to\infty} S_n$ is called the sum of the series.

$\boxed{\text{BQ}}$ **The Basic Question About Infinite Series**

How can we tell whether or not $\sum_{n=1}^{\infty} a_n$ converges by just looking at the sequence of terms $\{a_n\}_{n=1}^{\infty}$?

$\boxed{\text{SQ}}$ **Secondary Question**

If $\sum_{n=1}^{\infty} a_n$ converges then what does its sum equal?

Comment In sections 11.3-11.7 we will get very good at answering $\boxed{\text{BQ}}$ but $\boxed{\text{SQ}}$ is much more difficult.

This is one reason why geometric series are so special!

(In the second part of chapter 11, we'll discover more examples where $\boxed{\text{SQ}}$ can be answered.)

For constants a and r,

Geometric Series, $\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots$

(i) When $|r|<1$ the geometric series converges and

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}.$$

(ii) When $|r| \geq 1$, $\sum_{n=1}^{\infty} ar^{n-1}$ diverges.

Since a is a constant we have

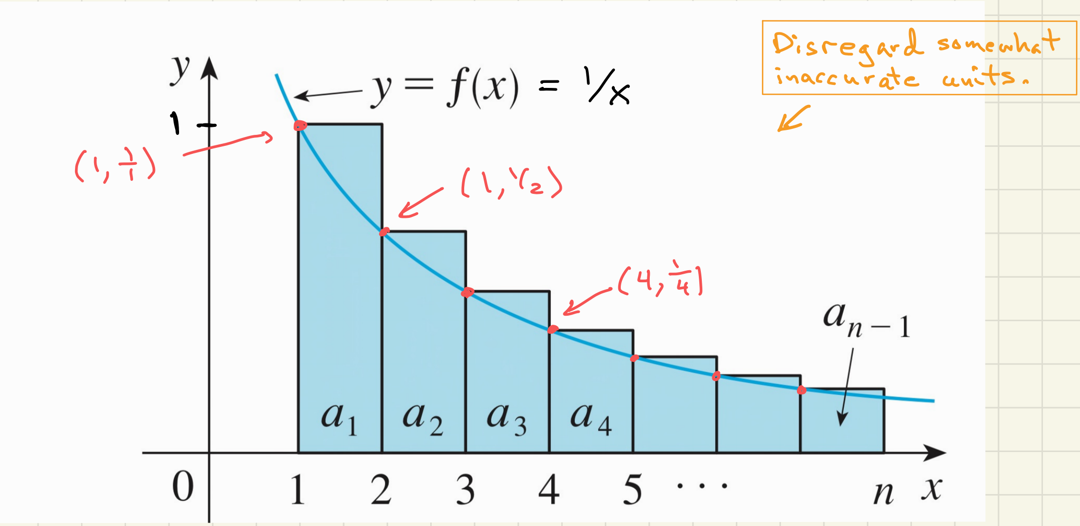
$$\sum_{n=1}^{\infty} a \cdot r^{n-1} = a \sum_{n=1}^{\infty} r^{n-1}$$

So for geometic series you could just remember that $\sum_{n=1}^{\infty} r^{n-1} = \frac{1}{1-r}$ if $|r|<1$

Because the indexing with the geometric series can sometimes be confusing, it may be best to remember it in the form:

$$\boxed{1 + r + r^2 + r^3 + r^4 + \cdots = \frac{1}{1-r} \quad \text{if } -1 < r < 1.}$$

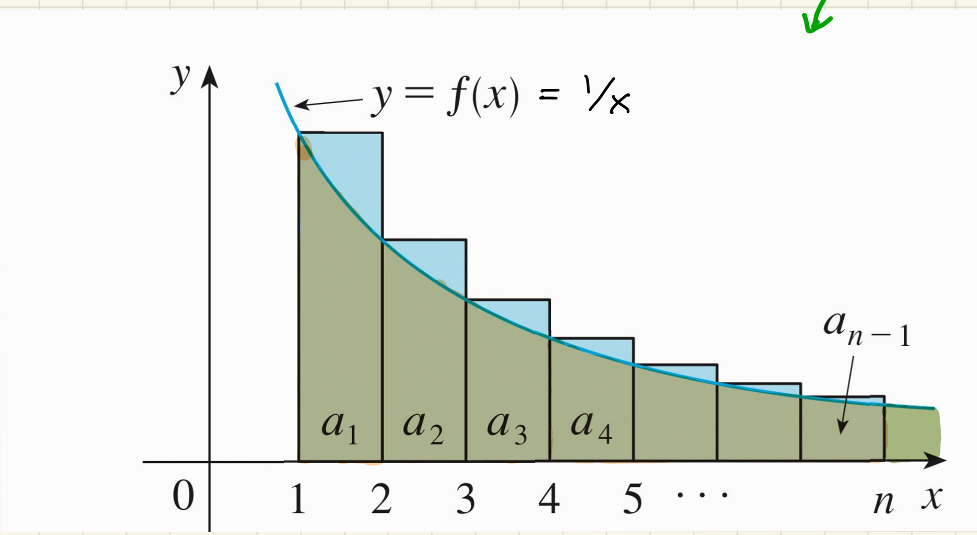
EXAMPLE The series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (called harmonic series)

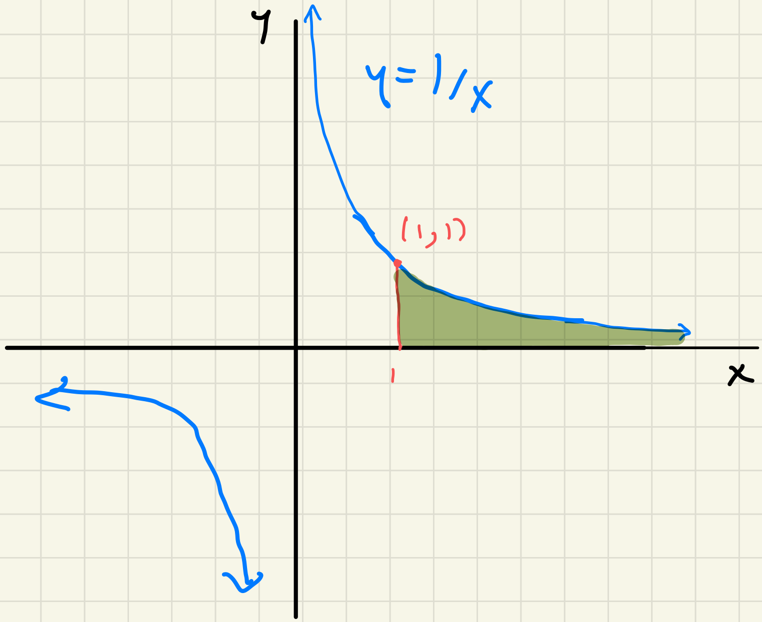


$a_1 = 1,\ a_2 = 1/2,\ a_3 = 1/3,\ \ldots,\ a_n = 1/n$

$$\text{Area of blue shaded region} = a_1 + a_2 + a_3 + \cdots + a_{n-1} = \sum_{i=1}^{n-1} \frac{1}{i} = S_{n-1}$$

$$\sum_{n=1}^{\infty} \frac{1}{n} = \sum_{n=1}^{\infty} a_n = \text{Area of infinitely many rectangles} > \text{Area of green shaded region}$$





Area of green shaded area $= \displaystyle\int_1^\infty \frac{1}{x}\,dx$

$$= \lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx = \lim_{b\to\infty} \ln(x)\Big|_{x=1}^{b} = \lim_{b\to\infty} \ln(b) = \infty$$

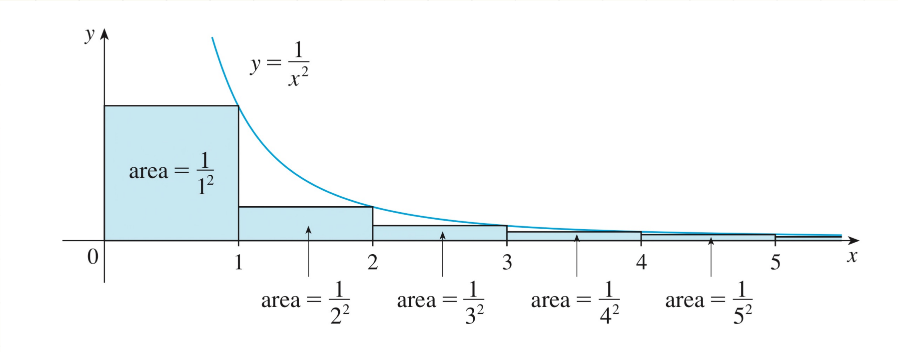
<u>Conclude</u>: $\displaystyle\sum_{n=1}^\infty \frac{1}{n} > \int_1^\infty \frac{1}{x}\,dx = +\infty$

So $\displaystyle\sum_{n=1}^\infty \frac{1}{n} = \infty$ and $\displaystyle\sum_{n=1}^\infty \frac{1}{n}$ diverges.

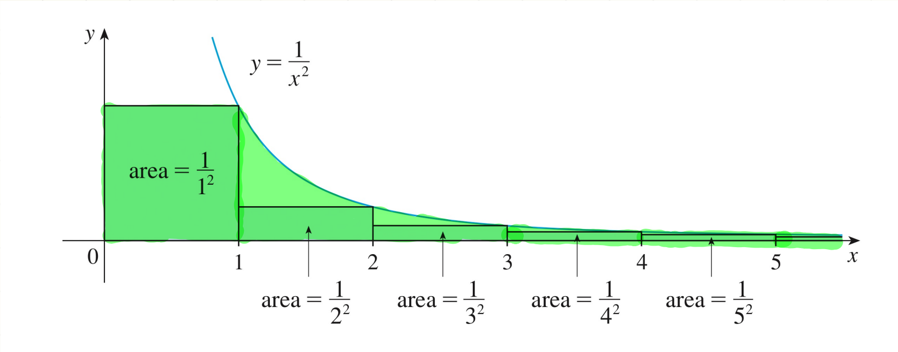
This is a special case of a more general result called the <u>Integral Test</u>

The Integral Test can also be used to show that some series converge.

Example $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$



Sum of areas of blue rectangles $= \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$



Area of green shaded region $= 1 + \displaystyle\int_1^{\infty} \frac{1}{x^2}\,dx \overset{*}{=} 1 + 1 = 2$

Conclude: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} < 2$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

$* \displaystyle\int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{b\to\infty} \int_1^{b} \frac{1}{x^2}\,dx = \lim_{b\to\infty} -\frac{1}{x}\Big|_{x=1}^{b} = \lim_{b\to\infty} 1 - \frac{1}{b} = 1$

from Stewart, Section 11.3

IT: **The Integral Test** Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)\,dx$ is convergent. In other words:

(i) If $\displaystyle\int_1^{\infty} f(x)\,dx$ is convergent, then $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent.

(ii) If $\displaystyle\int_1^{\infty} f(x)\,dx$ is divergent, then $\displaystyle\sum_{n=1}^{\infty} a_n$ is divergent.

The p-series (Important example using Integral Test)

Let $f(x) = \dfrac{1}{x^p}$ where $p > 0$, $p \neq 1$. Then

$$\int_1^{\infty} f(x)\,dx = \lim_{b\to\infty} \int_1^b x^{-p}\,dx = \lim_{b\to\infty} \left.\frac{x^{1-p}}{1-p}\right|_{x=1}^{b}$$

$p \neq 1$

$$= \lim_{b\to\infty} \frac{1}{b^{p-1}} - \frac{1}{1-p} = \begin{cases} \infty & \text{if } p<1 \\ \dfrac{1}{p-1} & \text{if } p>1 \end{cases}$$

$\boxed{\text{For } p>1,\ \lim_{b\to\infty} \frac{1}{b^{p-1}} = 0}$

Conclude

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ = Converges when $p > 1$ and diverges when $p \le 1$.

Examples:

- $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{.9}}$ diverge.
- $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1.1}}$ converge.

NOTE: The function $f(x) = \frac{1}{x^p}$, $p>1$ is continuous, positive and strictly decreasing for $x \ge 1$, so Integral Test applies.

Comment: while the Integral Test can be very useful to answer $\boxed{BQ}$ for some series $\sum_{n=1}^{\infty} a_n$ it has two significant drawbacks:

- The function $f(x)$ must satisfy the "hypotheses":

$$\boxed{f \text{ is continuous, positive and decreasing for } x \geq 1.}$$

- You need to be able to actually work the integral $\int_1^b f(x)\,dx$, but calculating integrals is often difficult or impossible!!