some algebra review

On Exam 2 it was necessary to solve the equation:

$$x^4 = 2x^2$$

The best way to do this is to express the equation as

$$x^4 - 2x^2 = 0$$

Then factor LHS to get

$$x^2(x+\sqrt{2})(x-\sqrt{2}) = 0$$

and the only way this can be true is if one of the factors x^2, $x+\sqrt{2}$ or $x-\sqrt{2}$ is 0. This gives solution:

$$x=0, \; x=-\sqrt{2} \text{ or } x=\sqrt{2}$$

An incorrect solution is

$$x^4 = 2x^2 \Rightarrow x^2 = 2 \Rightarrow x = \pm\sqrt{2}$$

(Clearly this is incorrect because $x=0$ is a solution to $x^4 = 2x^2$.)

The mistake here occurs in writing

$$x^4 = 2x^2 \Rightarrow x^2 = 2$$

which is incorrect because dividing by x^2 is illegitimate when $x=0$. This can be fixed by writing $x^4 = 2x^2 \Rightarrow x^2 = 2$ or $x=0$.

(and $x^2=2$ or $x=0 \Rightarrow x=-\sqrt{2}, x=\sqrt{2}$ or $x=0$.)

Sections 11.3 – 11.7 are focused on:

BQ The Basic Question About Infinite Series

How can we tell whether or not $\sum_{n=1}^{\infty} a_n$ converges by just looking at the sequence of terms $\{a_n\}_{n=1}^{\infty}$?

reminder If $\sum_{n=1}^{\infty} a_n$ converges or diverges then so does $\sum_{n=2}^{\infty} a_n$ or $\sum_{n=101}^{\infty} a_n$, etc. (In other words, where the indexing starts has no bearing on whether the series or diverges. For this reason we may phrase the basic question as

BQ The Basic Question About Infinite Series

How can we tell whether or not $\sum a_n$ converges by just looking at its sequence of terms $\{a_n\}$?

Geometric Series Fact:

$$\boxed{1+r+r^2+r^3+r^4+\cdots = \frac{1}{1-r} \quad \text{if } -1<r<1.}$$

One more geometric series example:

$$\sum_{n=1}^{\infty} \frac{3}{10^n} = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \frac{3}{10^4} + \cdots$$

Some partial sums are:

$s_1 = \frac{3}{10} = .3$, $s_2 = \frac{3}{10} + \frac{3}{100} = .33$, $s_3 = s_2 + \frac{3}{1000} = .333$, $s_4 = .3333$

and $s_n = .333\cdots3$ (repeated n times)

Conclude: $\displaystyle\sum_{n=1}^{\infty} \frac{3}{10^n} = \lim_{n\to\infty} s_n = .333\overline{3} = 1/3$

"repeating decimal"

Now let's view this as a geometric series:

$$\sum_{n=1}^{\infty} \frac{3}{10^n} = 3\sum_{n=1}^{\infty}\left(\frac{1}{10}\right)^n = 3\left(\frac{1}{10} + \left(\frac{1}{10}\right)^2 + \left(\frac{1}{10}\right)^3 + \cdots\right)$$

$$= 3\left(1 + \frac{1}{10} + \left(\frac{1}{10}\right)^2 + \left(\frac{1}{10}\right)^3 + \cdots\right) - 3\cdot 1$$

$$= 3\,\frac{1}{1-\frac{1}{10}} - 3 = 3\cdot\frac{10}{9} - 3 = \frac{10}{3} - \frac{9}{3} = \frac{1}{3}$$

Conclusion

Repeating decimals are examples of geometric series!

The Comparison Test Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

(i) If $\sum b_n$ is convergent and $a_n \leqslant b_n$ for all n, then $\sum a_n$ is also convergent.

(ii) If $\sum b_n$ is divergent and $a_n \geqslant b_n$ for all n, then $\sum a_n$ is also divergent.

<u>Example</u> $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+4^n}$. Does it converge or diverge?

To use comparison test, we take $\sum a_n = \sum \frac{1}{n+4^n}$.

Then we must choose a series $\sum b_n$ to compare with. Very often we'll choose $\sum b_n$ to be a geometric series or a p-series because then we know about the convergence or divergence of $\sum b_n$.

Here a good choice would be $\sum b_n = \sum \frac{1}{4^n}$ which converges because it's a geometric series with $r = \frac{1}{4} < 1$.

Then:

$$a_n = \frac{1}{n+4^n} \leq \frac{1}{4^n} = b_n \quad , \text{ for } n \geq 1$$

<u>Conclude</u> : $\sum a_n = \sum \frac{1}{n+4^n}$ converges

using part (i) of the Comparison Test.

LCT:

The Limit Comparison Test Suppose that Σa_n and Σb_n are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$, then either both series converge or both diverge.

examples:

① $\sum \frac{n}{n^3-1}$. Take $a_n = \frac{n}{n^3-1}$ and $b_n = \frac{1}{n^2}$. Then

$$\frac{a_n}{b_n} = \frac{n/n^3-1}{1/n^2} = \frac{n^3}{n^3-1} = \frac{1}{1-\frac{1}{n^3}} \xrightarrow[n\to\infty]{} 1 \; (= c)$$

since 1 is a finite number > 0, $\sum \frac{n}{n^3-1}$ converges.

② $\sum \frac{3\sqrt{n}+5}{n-\sqrt{n}+1}$ diverges.

$$\left.\begin{aligned} a_n &= \frac{3\sqrt{n}+5}{n-\sqrt{n}+1} \\ b_n &= \frac{1}{\sqrt{n}} \end{aligned}\right\} \Rightarrow \frac{a_n}{b_n} = \frac{3\sqrt{n}+5}{n-\sqrt{n}+1} \cdot \frac{\sqrt{n}}{1} = \frac{3n+5\sqrt{n}}{n-\sqrt{n}+1} \xrightarrow[n\to\infty]{} 3$$

and $\sum \frac{1}{\sqrt{n}}$ diverges because it is p-series with $p = \frac{1}{2} \leq 1$.

$$\boxed{\frac{3n+5n^{1/2}}{n-n^{1/2}+1} \cdot \frac{1/n}{1/n} = \frac{3+5\frac{1}{n^{1/2}}}{1-\frac{1}{n^{1/2}}+\frac{1}{n}} \xrightarrow[n\to\infty]{} \frac{3+0}{1-0+0} = 3}$$