The Ratio Test Let $\sum a_n$ be a positive series and
$$L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}.$$

(i) If $L<1$ then $\sum a_n$ converges.

(ii) If $L>1$ then $\sum a_n$ diverges.

(iii) If $L=1$ this test is inconclusive.

There is a variant of the Ratio Test that applies to series $\sum a_n$ which are not necessarily positive:

Addendum to Ratio Test

If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$ where $L>1$ then the series $\sum a_n$ diverges.

Example: The series $\sum_{n=0}^{\infty} \frac{n(-2)^n}{3n^2+1}$ diverges.

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)(-2)^{n+1}}{3(n+1)^2+1} \cdot \frac{3n^2+1}{n(-2)^n} = -2\,\frac{n+1}{n}\,\frac{3n^2+1}{3n^2+6n+4}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = 2\,\frac{n+1}{n}\,\frac{3n^2+1}{3n^2+6n+4} \xrightarrow[n\to\infty]{} 2$$

Since $2>1$, the series diverges.

The Root Test is a close relative of the Ratio Test:

The Root Test

(i) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(ii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, the Root Test is inconclusive.

<u>Example</u>

The series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{2n}}$ converges absolutely.

$$a_n = \frac{(-1)^{n-1}}{n^{2n}}, \quad |a_n| = \frac{1}{n^{2n}}$$

$$\sqrt[n]{|a_n|} = \left(\frac{1}{n^{2n}}\right)^{1/n} = \left(n^{-2n}\right)^{1/n} = n^{-2}$$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{1}{n^2} = 0 = L$$

Absolutely Convergent

Summary of Results to Answer the Basic Question $\boxed{BQ}$.

general:

- Test for Divergence — DT
- Linearity — Linear

important examples:

- Geometric Series — GS
- p-series — PS

positive series:

- Integral Test — IT
- Comparison Test — CT
- Limit Comparison Test — LCT
- Ratio Test — Ratio
- Root Test — Root

series with positive and negative terms:

- Absolute Convergence Theorem — Abs
- Alternating Series Test — AST
- Addendum to Ratio Test — Ratio

Some Comments on next page.

COMMENTS:

1) "Linearity" has two parts:

- If c is a constant then $\sum_{n=1}^{\infty} c \cdot a_n = c \sum_{n=1}^{\infty} a_n$

  (Here c being a constant means that it does not depend on the index variable n. We could say "c has no n's in it".)

- $\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$ if both series converge.

2) In the two comparison tests, $\sum a_n$ is the series we are interested in determining convergence/divergence for. We must choose a series $\sum b_n$ to compare with. This series $\sum b_n$ needs to be one that we already know whether it converges or diverges. Most of the time that means we will choose $\sum b_n$ to be either a geometric series or a p-series.

3) Remember the terminology for absolute convergence and conditional convergence....

4) Examples can be helpful to remember fine points!!

- Geometric Series $\sum r^n$ converges only when $|r| < 1$.
- p-Series $\sum \frac{1}{n^p}$ converges only when $p > 1$
- $\sum \frac{(-1)^n}{n}$ converges but $\sum \frac{1}{n}$ diverges
- The ratio test doesn't work for p-series but it does work for geometric series.

Power Series

$$= \sum_{n=0}^{\infty} a_n \quad \text{where } a_n = c_n (x-a)^n$$

A series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

is called a power series centered at $x=a$. Here a is a fixed number, x is a variable, and each c_n is a number that does not depend on x.

Most often we consider the case where $a=0$

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

(So think of this as a "polynomial with infinite degree".)

A power series may converge for some values of x and diverge for other values of x. Think of it as describing a function (with variable x)

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

whose domain consists of all numbers x for which the series converges.

Let's examine the example

$$f(x) = \sum_{n=0}^{\infty} n^2 x^n = 0 + x + 4x^2 + 9x^3 + 16x^4 + \cdots$$

For what values of x does the power series converge?

To answer we'll use Ratio Test as a start.

$$\frac{|a_{n+1}|}{|a_n|} = \frac{|(n+1)^2 x^{n+1}|}{|n^2 x^n|} = \frac{(n+1)^2}{n^2} \frac{|x|^{\cancel{n+1}}}{\cancel{|x|^n}} = \frac{n^2+2n+1}{n^2}|x|$$

The variable x can be positive or negative.

So $\displaystyle\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = |x|. = L$

The ratio test says that the power series $\displaystyle\sum_{n=0}^{\infty} n^2 x^n$ converges when $|x|<1$ (that is, $-1<x<1$) and diverges when $|x|>1$ (that is, $x<-1$ or $x>1$).

So the domain of $f(x) = \displaystyle\sum_{n=0}^{\infty} c_n x^n$ is one of four possible intervals:

$$(-1, 1), \ [-1, 1), \ (-1, 1], \text{ or } [-1, 1]$$

Which one? In this example, its $\text{domain}(f) = (-1, 1)$ because ....

Examine the two endpoints $x=-1$ and $x=1$ one at a time.

$$f(x) = \sum_{n=0}^{\infty} n^2 x^n$$

<u>$x=-1$</u> $\quad \sum_{n=0}^{\infty} n^2 (-1)^n$

This series diverges because $\lim_{n\to\infty} n^2(-1)^n = DNE$ (Test for Divergence)

<u>$x=1$</u> $\quad \sum_{n=0}^{\infty} n^2$

This series also diverges b/c $\lim_{n\to\infty} n^2 = \infty$.

<u>Conclude</u> Neither $x=-1$ nor $x=1$ is in the domain of $f(x)$.

Therefore $\text{Domain}(f) = (-1, 1)$

$$\| $$

$\{x \text{ where } -1 < x < 1\}$

Infinite Series Review Sheet: Convergence Tests

<u>Test for Divergence</u>. *If* $\lim_{n\to\infty} a_n \neq 0$ *then the series* $\sum_{n=1}^{\infty} a_n$ *diverges.*

<u>Linearity</u>. *If* c *is a constant and the series* $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *converge then* $\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} a + n + b_n = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$.

<u>Geometric Series</u>. *If* r *is a constant then* $\sum_{n=0}^{\infty} r^n$ *converges when* $|r| < 1$ *and diverges when* $|r| \geq 1$. *When* $|r| < 1$, *the sum of this series equals* $1/(1-r)$.

<u>p-Series</u>. *If* p *is a constant then* $\sum n = 1^{\infty} \frac{1}{n^p}$ *converges when* $p > 1$ *and diverges when* $p \leq 1$.

<u>Integral Test</u>. *Let* $f(x)$ *be a positive, continuous decreasing function for* $x \geq 1$ *and let* $a_n = f(n)$.
(a) If the improper integral $\int_1^{\infty} f(x)\,dx$ *converges then the series* $\sum_{n=1}^{\infty} a_n$ *converges.*
(b) If the improper integral $\int_1^{\infty} f(x)\,dx$ *diverges then the series* $\sum_{n=1}^{\infty} a_n$ *diverges.*

<u>Comparison Test</u>. *Let* $\{a_n\}$ *and* $\{b_n\}$ *be sequences of positive numbers with* $a_n \leq b_n$ *for all positive integers* n.
(a) If $\sum_{n=1}^{\infty} b_n$ *converges then* $\sum_{n=1}^{\infty} a_n$ *converges.*
(b) If $\sum_{n=1}^{\infty} a_n$ *diverges then* $\sum_{n=1}^{\infty} b_n$ *diverges.*

<u>Limit Comparison Test</u>. *Let* $\{a_n\}$ *and* $\{b_n\}$ *be sequences of positive numbers with* $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = c$.
(a) If the series $\sum_{n=1}^{\infty} b_n$ *converges and* $0 \leq c < \infty$ *then* $\sum_{n=1}^{\infty} a_n$ *converges.*
(b) If the series $\sum_{n=1}^{\infty} b_n$ *diverges and* $0 < c \leq \infty$ *then* $\sum_{n=1}^{\infty} a_n$ *diverges.*

<u>Ratio Test</u>. *Suppose that* $a_n > 0$ *for all* n *and that* $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$.
(a) If $L < 1$ *then* $\sum_{n=1}^{\infty} a_n$ *converges.*
(b) If $L > 1$ *then* $\sum_{n=1}^{\infty} a_n$ *diverges. In fact, if* $L > 1$ *then* $\lim_{n\to\infty} a_n = \infty$.

<u>Root Test</u>. *Suppose that* $a_n > 0$ *for all* n *and that* $\lim_{n\to\infty} \sqrt[n]{a_n} = L$.
(a) If $L < 1$ *then* $\sum_{n=1}^{\infty} a_n$ *converges.*
(b) If $L > 1$ *then* $\sum_{n=1}^{\infty} a_n$ *diverges .*

<u>Alternating Series Test</u>. *Let* $\{b_n\}$ *be a decreasing sequence with* $\lim_{n\to\infty} b_n = 0$ *and* $b_n > 0$. *Then the series* $\sum_{n=1}^{\infty} (-1)^n b_n$ *converges.*

<u>Absolute Convergence Test</u>. *If* $\sum_{n=1}^{\infty} |a_n|$ *converges (that is, if* $\sum_{n=1}^{\infty} a_n$ *"converges absolutely") then* $\sum_{n=1}^{\infty} a_n$ *converges.*

<u>Addendum to Ratio Test</u>. *If* $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$ *and* $L > 1$ *then* $\sum_{n=1}^{\infty} a_n$ *diverges.*

A series $\sum_{n=1}^{\infty} a_n$ is said to **converge absolutely** if the positive series $\sum_{n=1}^{\infty} |a_n|$ converges.
Note that the Integral, Comparison, Limit Comparison, Root and Ratio Tests are all tests that apply only to positive series (or really to any series that has only finitely many negative terms). However they can be used to determine the absolute convergence of any series.

IMPORTANT BASIC PRINCIPLE: An infinite series $\sum_{n=M}^{\infty} a_n$ will converge if and only if the series $\sum_{n=L}^{\infty} a_n$ converges. This means that the value of the starting index for the series has no effect on whether it converges or diverges. So it is common to leave off the indexing entirely and just say that $\sum a_n$ converges or diverges. (The same comments apply in like manner for absolute convergence and conditional convergence.) However, if you want to determine the sum of a convergent series then that does depend on where the indexing starts. For example, $\sum_{n=0}^{\infty}(2/3)^n = 3$ but $\sum_{n=1}^{\infty}(2/3)^n = 2$.
