A series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

is a "power series centered at $x=a$". (Here a is a constant, x is a variable, and each c_n is a number that does not depend on x.) When $a=0$ we have

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

Each power series determines a function

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

The domain of this function consists of all numbers x for which the series converges. This domain is always an interval centered at $x=a$, and it is called the <u>interval of convergence</u> of the power series.

<u>Example</u> Geometric Series!!

$$f(x) = \sum_{n=0}^{\infty} x^n$$

(power series where $c_n = 1$ for all n)

This function has $\text{domain}(f) = \{x \text{ where } |x| < 1\} = (-1, 1)$ and we have a formula

$$f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad , \quad -1 < x < 1$$

Some Examples of Power Series

1) $\sum_{n=0}^{\infty} n^2(x-1)^n = (x-1) + 4(x-1)^2 + 9(x-1)^3 + \cdots$ $a=1$, $c_n = n^2$

2) $\sum_{n=0}^{\infty} n^2(x-\pi)^n = (x-\pi) + 4(x-\pi)^2 + 9(x-\pi)^3$

3) $\sum_{n=0}^{\infty} n^2 x^n = 0 + x + 4x^2 + 9x^3 + 16x^4 + \cdots$

4) $\sum_{n=0}^{\infty} n^2 x^{n+5} = x^5 \sum_{n=0}^{\infty} n^2 x^n$

5) $\sum_{n=0}^{\infty} (-1)^{n-1} n^2 x^n = 0 + x - 4x^2 + 9x^3 - 16x^4 + \cdots$

6) $\sum_{n\ 0}^{\infty} n^2 x^{2n} = x^2 + 4x^4 + 9x^6 + 16x^8 + \cdots$

7) $\sum_{n=0}^{\infty} n^2 x^{3n+1} = x^4 + 4x^7 + 9x^{10} + 16x^{13} + \cdots$

For (6) If we write $\sum_{n=0}^{\infty} n^2 x^{2n}$ as $\sum_{n=0}^{\infty} c_n x^n$ then $c_n = 0$ for odd values of n. In fact

$$c_n = \begin{cases} 0 & \text{when } n \text{ is odd} \\ \left(\frac{n}{2}\right)^2 & \text{when } n \text{ is even} \end{cases}$$

For (7) If we write $\sum_{n=0}^{\infty} n^2 x^{3n+1} = \sum_{n=0}^{\infty} c_n x^n$ then $c_n = 0$ whenever n has a remainder of 0 or 2 when divided by 3

Interval of Convergence I

Every power series

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

has an associated radius of convergence R with the property that the series

- converges when $|x-a| < R$, and
- diverges when $|x-a| > R$.

This means that the interval of convergence I of the power series is one of:

$$(a-R, a+R), \quad [a-R, a+R), \quad (a-R, a+R], \quad [a-R, a+R]$$

open interval — half-open interval — closed interval

Notice that each of these possible intervals has center (that is, midpoint) at $x=a$.

Strategy for determining interval of convergence:

(i) Use Ratio Test to find R.

(ii) Use other tests to determine convergence at the two endpoints: $x = a-R$, $x = a+R$

(iii) In the cases where $R=0$ or $R=\infty$, step (2) will be unnecessary.

Examples

① $\sum_{n=0}^{\infty} x^n$, ✓ $R=1$, $I=(-1,1)$ ✓ (geometric)

② $\sum_{n=0}^{\infty} \frac{3^n}{n} x^n$. ⓘ Use Ratio test: $a_n = \frac{3^n}{n} x^n$

$$\frac{|a_{n+1}|}{|a_n|} = \frac{3^{n+1}|x|^{n+1}}{n+1} \cdot \frac{n}{3^n |x|^n} = 3|x| \frac{1}{1+1/n} \xrightarrow[n\to\infty]{} 3|x| = L$$

So, series converges when $3|x|<1$ (means $|x|<1/3$)
and diverges when $3|x|>1$ (means $|x|>1/3$) $R=\frac{1}{3}$

(ii) The endpoints are $x=-1/3$ and $x=1/3$.

check $x=\frac{1}{3}$: $\sum_{n=0}^{\infty} \frac{3^n}{n}\left(\frac{1}{3}\right)^n = \sum_{n=0}^{\infty} \frac{1}{n}$ diverges. PS

check $x=-\frac{1}{3}$: $\sum_{n=0}^{\infty} \frac{3^n}{n}\left(-\frac{1}{3}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{n}$ converges. AST

conclude: $I=\left[-\frac{1}{3}, \frac{1}{3}\right)$, ✓ $R=1/3$

③ $\sum_{n=0}^{\infty} \frac{n!}{3^n} x^n$.

$\boxed{\frac{(n+1)!}{n!} = \frac{(n+1)n!}{n!} = n+1}$

never less than 1

$$\frac{|a_{n+1}|}{|a_n|} = \frac{(n+1)!}{3^{n+1}|x|^{n+1}} \frac{3^n|x|^{n+1}}{n!} = \frac{n+1}{3}|x| \xrightarrow[n\to\infty]{} \infty$$

if $x \neq 0$

conclude The series always diverges when $x \neq 0$.

So $R=0$, $I=\{0\}$

case (iii) applies here.

So power series converges only when $x=0$.

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