Lines in 3-space

Each line ℓ in xyz-space has a parametrization:

$$\ell: \begin{cases} x = at + x_0 \\ y = bt + y_0 \\ z = ct + z_0 \end{cases}$$

$P = (x_0, y_0, z_0)$ is a point on ℓ $t=0$

for some constants a, b, c and x_0, y_0, z_0.

We can also express these equations in vector form as

$$\ell: \vec{r}(t) = t\vec{d} + \vec{r}_0$$

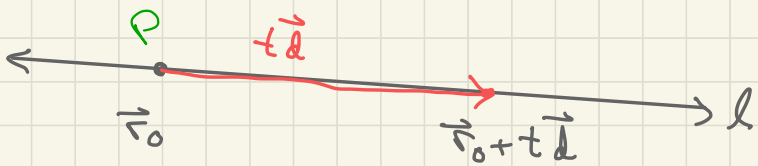
where

$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle = \langle at + x_0, bt + y_0, ct + z_0 \rangle$

$\vec{d} = \langle a, b, c \rangle$ = direction vector for ℓ ← a vector parallel to ℓ

$\vec{r}_0 = \langle x_0, y_0, z_0 \rangle$ = point on ℓ

picture



Very Special Examples:

A point on x-axis has coordinates $(x, 0, 0)$

x-axis: $\begin{cases} x = t = 1 \cdot t + 0 \\ y = 0 = 0 \cdot t + 0 \\ z = 0 = 0 \cdot t + 0 \end{cases}$ or $\vec{r}(t) = \langle t, 0, 0 \rangle = t\vec{i}$

y-axis: $\vec{r}(t) = t\vec{j}$ ← $\begin{cases} x = 0 \\ y = t \\ z = 0 \end{cases}$

z-axis: $\vec{r}(t) = t\vec{k}$

Two lines in 3-space are either:

1. equal
2. parallel but not equal
3. intersect in one point
4. skew (means 1, 2, 3 do not hold)

Example Do ℓ_1 and ℓ_2 intersect? If so, where?

$$\ell_1: \begin{cases} x = t-2 \\ y = t-4 \\ z = -2t+6 \end{cases} \quad \text{and} \quad \ell_2: \begin{cases} x = -4t+5 \\ y = -1 = 0t-1 \\ z = 10t-10 \end{cases}$$

Note first: Direction vectors for ℓ_1 and ℓ_2 are $\vec{d}_1 = \langle 1, 1, -2 \rangle$ and $\vec{d}_2 = \langle -4, 0, 10 \rangle$, which are not parallel. So either (3) or (4) hold.

Look for point of intersection:

$$\begin{cases} t-2 = -4s+5 \\ t-4 = -1 \\ -2t+6 = 10s-10 \end{cases}$$

Solve simultaneously?

Here $t = 4-1 = 3$ (2nd equation), $10s-10 = -2(3)+6 = 0$ (3rd equation) $\Rightarrow s=1$, and $(t=3, s=1)$ solves 1st equation.

Conclude: The lines intersect at point P on ℓ_1 where $t=3$. $P = (3-2, 3-4, -2(3)+6) = (1, -1, 0)$

example

Find equations for the line containing points $P=(-1,3,4)$ and $Q=(2,2,-1)$.

Strategy First identify (i) a point on the line, and (ii) a direction vector for the line.

Always use this strategy to find equations for a line!

Here (i) $Q=(2,2,-1)$ is a point on ℓ

(ii) The vector $\vec{d}=\overrightarrow{PQ}=\langle 3,-1,-5\rangle$ is a direction vector for ℓ.

so we get parametric equations for ℓ:

$$\ell:\begin{cases} x=3t+2 \\ y=-t+2 \\ z=-5t-1 \end{cases}$$

OR in vector form

$$\ell:\ \vec{r}(t)=\langle 3t+2,\ -t+2,\ 5t-1\rangle$$

comment: For (i) we might have chosen instead the point $P=(-1,3,4)$ getting

$$\ell:\begin{cases} x=3t-1 \\ y=-t+3 \\ z=-5t+4 \end{cases}$$

← For $t=1$, $(x,y,z)=(2,2,-1)=Q$

which is a different parametrization for the same line ℓ!

example A line ℓ is perpendicular to lines ℓ_1 and ℓ_2 and passes thru their intersection point.

$$\ell_1: \begin{cases} x = t-2 \\ y = t-4 \\ z = -2t+6 \end{cases} \quad \text{and} \quad \ell_2: \begin{cases} x = -4t+5 \\ y = -1 \\ z = 10t-10 \end{cases}$$

Find equations for ℓ.

strategy from previous page

(i) P = point of intersection of ℓ_1 and ℓ_2 a point on ℓ.

(ii) If $\vec{d}_1$ and $\vec{d}_2$ are direction vectors for ℓ_1 and ℓ_2 respectively then $\vec{d}_1 \times \vec{d}_2$ is a direction vector for ℓ.

(i) P = point of intersection = point on ℓ_1 with $t=3$
$= (1,-1,0)$ (worked in a previous example)

(ii) $\vec{d}_1 = \langle 1,1,-2 \rangle$ and $\vec{d}_2 = \langle -4,0,10 \rangle$

$$\vec{d}_1 \times \vec{d}_2 = \langle 1,1,-2 \rangle \times \langle -4,0,10 \rangle = \langle 10,-2,4 \rangle$$

calculation on next page

Parametric Equations

$$\ell: \begin{cases} x = 10t+1 \\ y = -2t-1 \\ z = 4t \end{cases}$$

Vector Equation for ℓ:

$$\ell: \vec{r}(t) = \langle 10,-2,4 \rangle t + \langle 1,-1,0 \rangle$$
$$= \langle 10t+1, -2t-1, 4t \rangle$$

$$\langle 1,1,-2\rangle \times \langle -4,0,10\rangle = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & -2 \\ -4 & 0 & 10 \end{vmatrix}$$

$$= (1\cdot 10 - 0\cdot(-2))\vec{i} - (1\cdot 10 - (-2)(-4))\vec{j} + (1\cdot 0 - 1\cdot(-4))\vec{k}$$

$$= 10\vec{i} - 2\vec{j} + 4\vec{k}$$

check: $\langle 1,1,-2\rangle \cdot \langle 10,-2,4\rangle$

$\langle 1,1,-2\rangle \cdot \langle 10,-2,4\rangle = 10 - 2 - 8 = 0$

$\langle -4,0,10\rangle \cdot \langle 10,-2,4\rangle = 0$

Planes in 3-space

Every plane p in 3-space has an equation of the form:

$$p: Ax + By + Cz + D = 0$$

(called a "linear equation with 3 variables".)

Some comments

① A point $P = (x_0, y_0, z_0)$ is on the plane p whenever the equation $Ax_0 + By_0 + Cz_0 + D = 0$ holds true.

② The vector $\vec{N} = \langle A, B, C \rangle$ is perpendicular to p. We say, $\vec{N}$ is a <u>normal vector</u> for p.

<u>explain</u>: Suppose $P = (x_0, y_0, z_0)$ and $Q = (x_1, y_1, z_1)$ are two points on p. Then $Ax_0 + By_0 + Cz_0 + D = 0$ and $Ax_1 + By_1 + Cz_1 + D = 0$.

$$\begin{aligned}\vec{PQ} \cdot \vec{N} &= \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle \cdot \langle A, B, C \rangle \\ &= A(x_1 - x_0) + B(y_1 - y_0) + C(z_1 - z_0) \\ &= (Ax_1 + By_1 + Cz_1) - (Ax_0 + By_0 + Cz_0) \\ &= (-D) - (-D) = 0\end{aligned}$$

This shows that every vector in p is perpendicular to $\vec{N}$.

Example Consider the plane $p: 3x-2y-z+6=0$.

① Is the point $(-1,-1,-1)$ on p? No

$3(-1)-2(-1)-(-1)+6 = 8 \neq 0$

② Find some points that are on p.

Look for intercepts with coordinate axes:

x-intercept: $(0,0,-2)$ (plug $y=0, z=0$ into equation)

$3x-2(0)-0+6=0 \Rightarrow x=-2$

y-intercept: $(0,3,0)$ (plug in $x=0, z=6$)

z-intercept: $(0,0,6)$ (plug in $x=0, y=8$)

or use a hit or miss approach:

E.g. - Try $x=2, y=-1 \Rightarrow 3(2)-2(-1)-z+6=0 \Rightarrow z=14$

$\Rightarrow (2,-1,14)$ is on the plane p.

③ Equations for some planes parallel to p are:

$3x-2y-z-6=0$

$3x-2y-z+1=0$ ← those all have normal vector $\langle 3,-2,-1\rangle$. (Parallel planes must have parallel normal vectors)

$3x-2y-z-17=0$

$3x-2y-z=0$ ← This one goes thru origin!

④ How does the plane $p_1: -12x+8y+4z-24=0$ compare with p?

answer: $p=p_1$ (just multiply equation for p by -4)