Lines and Planes — RECAP

Lines in 3-space

Each line ℓ in xyz-space has a parametrization:

$$\ell : \begin{cases} x = at + x_0 \\ y = bt + y_0 \\ z = ct + z_0 \end{cases}$$

for some constants a, b, c and x_0, y_0, z_0. The vector $\vec{d}_\ell = \langle a, b, c \rangle$ is called a <u>direction vector</u> for ℓ (it is parallel to ℓ). The point on ℓ where $t=0$ is (x_0, y_0, z_0).

or write this as:

$$Ax + By + Cz = -D$$

Planes in 3-space

Every plane p in 3-space has an equation of the form:

$$p : Ax + By + Cz + D = 0$$

(called a "linear equation with 3 variables".)

The vector $\vec{N} = \langle A, B, C \rangle$ is perpendicular to p, it is called a <u>normal vector</u> for p.

Example: Are these two lines parallel?

$$\ell_1: \begin{cases} x = -2t + 0 \\ y = t + 0 \\ z = -3t + 0 \end{cases} \qquad \ell_2: \begin{cases} x = 6t + 4 \\ y = -3t - 2 \\ z = 9t + 6 \end{cases}$$

$\vec{d}_1$ = direction vector for ℓ_1 $= \langle -2, 1, -3 \rangle$ ← these are parallel vectors

$\vec{d}_2$ = direction vector for ℓ_2 $= \langle 6, -3, 9 \rangle = 3\vec{d}_1$

conclude ℓ_1 and ℓ_2 are parallel.

note well: If $\vec{d}$ is a direction vector for ℓ then any vector parallel to $\vec{d}$ is also a direction vector for ℓ. In this example $\langle -2, 1, -3 \rangle$ is a direction vector for both ℓ_1 and ℓ_2.

Follow-Up Question Does ℓ_2 equal ℓ_1?

$(0,0,0)$ is a point on ℓ_1. Is it also on ℓ_2? YES:

The equations $\begin{cases} 6t + 4 = 0 \\ -3t - 2 = 0 \\ 9t + 6 = 0 \end{cases}$ have a solution $t = -\frac{2}{3}$

conclude $\ell_2 = \ell$ (Parallel lines with a point in common must be equal.)

Planes in 3-space

Every plane p in 3-space has an equation of the form:

$$p: Ax + By + Cz + D = 0$$

(called a "linear equation with 3 variables".)

The vector $\vec{N} = \langle A, B, C \rangle$ is perpendicular to p, it is called a <u>normal vector</u> for p.

<u>note well:</u> If $\vec{N}$ is a normal vector for p then <u>any</u> vector parallel to $\vec{N}$ is also a normal vector for p.

continued from last class

<u>Example</u> Consider the plane $p: 3x-2y-z+6=0$.

$\tilde{N} = \langle 3,-2,-1 \rangle$

① Is the point $(-1,-1,-1)$ on p? <u>No</u>

② Find some points that are on p.

or use a hit or miss approach:

E.G. - Try $x=2, y=-1 \Rightarrow 3(2)-2(-1)-z+6=0 \Rightarrow z=14$
$\Rightarrow (2,-1,14)$ is on the plane p.

③ Equations for some planes parallel to p are:

$3x-2y-z-6=0$

$3x-2y-z+1=0$

$3x-2y-z-17=0$

these all have normal vector $\langle 3,-2,-1 \rangle$. (Parallel planes must have parallel normal vectors)

$3x-2y-z=0$ ← This one goes thru origin!

④ How does the plane $p_1: -12x+8y+4z-24=0$ compare with p?

<u>answer:</u> $p=p_1$ (just multiply equation for p by -4)

Strategy for finding the equation of a plane p

(i) Find a point $P = (x_0, y_0, z_0)$ on p.

(ii) Find a vector $\vec{N} = \langle A, B, C \rangle$ perpendicular to p ("normal vector").

Then an equation for the plane is

$$p: A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

explain If a point $Q = (x, y, z)$ is in the plane p

then $\vec{PQ} = \langle x - x_0, y - y_0, z - z_0 \rangle$ is $\perp$ to $\vec{N}$.

So $0 = \vec{N} \cdot \vec{PQ} = A(x - x_0) + B(y - y_0) + C(z - z_0)$.

example: A plane p is parallel to the plane $p_1 : 3x + 7y - 5z = 0$ and contains $(-1, -2, 3)$. Find an equation for p.

(i) $P = (-1, -2, 3)$ is on p

(ii) $\vec{N}_1 = \langle 3, 7, -5 \rangle$ is a normal vector for both p_1 and p.

So $p: 3(x + 1) + 7(y + 2) - 5(z - 3) = 0$

$p: 3x + 7y - 5z + 32 = 0$

Example:

Find an equation for the plane p which contains the three points $P=(1,-1,3)$, $Q=(5,5,5)$ and $R=(-2,0,4)$.

(i) $R=(-2,0,4)$ is a point on p

(ii) $\vec{PQ}\times\vec{PR}$ is a normal vector for p

$$\vec{PQ} = \langle 4, 6, 2\rangle$$
$$\vec{PR} = \langle -3, 1, 1\rangle$$

$$\vec{PQ}\times\vec{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & 6 & 2 \\ -3 & 1 & 1 \end{vmatrix} = 4\vec{i} - 10\vec{j} + 22\vec{k} = \vec{N}$$

$$p: 4(x+2) - 10(y-0) + 22(z-4) = 0$$
$$4x - 10y + 22z - 80 = 0$$
$$2x - 5y + 11z - 40 = 0$$

Two lines in 3-space are either:

1. equal
2. parallel but not equal
3. intersect in one point
4. skew

note: In cases 2 and 3 there is a unique plane containing the two lines.

Two planes in 3-space are either:

1. equal
2. parallel but not equal
3. intersect in a line

note: In 3, the cross product of normal vectors for the planes is a direction vector for the line of intersection.

A line ℓ and a plane p satisfy one of:

1. ℓ is contained in p
2. ℓ is parallel to p but not contained in p
3. ℓ intersects p in one point

note: In 1 or 2, a direction vector for ℓ will be perpendicular to a normal vector for p.