Lines and Planes — RECAP

Lines in 3-space

Each line ℓ in xyz-space has a parametrization:

$$\ell: \begin{cases} x = at + x_0 \\ y = bt + y_0 \\ z = ct + z_0 \end{cases}$$

for some constants a, b, c and x_0, y_0, z_0. The vector $\vec{d}_\ell = \langle a, b, c \rangle$ is called a <u>direction vector</u> for ℓ (it is parallel to ℓ). The point on ℓ where $t=0$ is (x_0, y_0, z_0).

or write this as:
$Ax + By + Cz = -D$

Planes in 3-space

Every plane p in 3-space has an equation of the form:

$$p: Ax + By + Cz + D = 0$$

(called a "linear equation with 3 variables".)

The vector $\vec{N} = \langle A, B, C \rangle$ is perpendicular to p, it is called a <u>normal vector</u> for p.

important info!!

Strategy for finding scalar equations for a line:

(i) Find a point $P = (x_0, y_0, z_0)$ on ℓ.

(ii) Find a vector $\vec{d}_\ell = \langle a, b, c \rangle$ parallel to ℓ ("direction vector")

Then equations for ℓ are:

$$\ell : \begin{cases} x = at + x_0 \\ y = bt + y_0 \\ z = ct + z_0 \end{cases}$$

Strategy for finding the equation of a plane p

(i) Find a point $P = (x_0, y_0, z_0)$ on p.

(ii) Find a vector $\vec{N}_p = \langle a, b, c \rangle$ perpendicular to p ("normal vector").

Then an equation for the plane is

$$p : A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

Example: $P=(-1,1,3)$, $p: 2x+7y-2z=56$

(a) Find equations for line ℓ thru P perpendicular to p.

(b) Find point Q where ℓ intersects p.

(c) Find the distance from P to Q.

(a) (i) $P=(-1,1,3)$

(ii) $\vec{d}_\ell = \vec{N}_p = \langle 2,7,-2\rangle$

$$\begin{cases} x = 2t-1 \\ y = 7t+1 \\ z = -2t+3 \end{cases}$$

Solve for t

(b) $2(2t-1)+7(7t+1)-2(-2t+3)=56$

$57t-1=56$, $57t=57$

$t=1$ So $Q=(1,8,1)$

(c) $\text{dist}(P,Q) = \sqrt{(1-(-1))^2+(8-1)^2+(1-3)^2}$

$= \sqrt{57}$

Interpretation

The point Q is the "foot of the perpendicular from P to p." It is the closest point to P on the plane p.

The distance from P to Q is the shortest distance from P to the plane.

<u>ASIDE</u> This approach can be used to find the distance from any point to any plane, as follows:

Distance from $P = (x_0, y_0, z_0)$ to $p: Ax + By + Cz = D$??

ⓐ $\ell : \begin{cases} x = At + x_0 \\ y = Bt + y_0 \\ z = Ct + z_0 \end{cases}$ is line $\perp$ to p thru P.

ⓑ $A(At + x_0) + B(Bt + y_0) + C(Ct + z_0) = D$

$\Rightarrow (A^2 + B^2 + C^2)t = D - (Ax_0 + By_0 + Cz_0)$

$\Rightarrow t = \dfrac{D - (Ax_0 + By_0 + Cz_0)}{A^2 + B^2 + C^2} = t_0$

for convenience just call this t_0

Q = intersection of ℓ and p

$= (At_0 + x_0, Bt_0 + y_0, Ct_0 + z_0)$

ⓒ $\text{dist}(P, Q) = \left((At_0)^2 + (Bt_0)^2 + (Ct_0)^2\right)^{1/2}$

$= (A^2 + B^2 + C^2)^{1/2} \, |t_0| = \dfrac{|Ax_0 + By_0 + Cz_0 - D|}{\sqrt{A^2 + B^2 + C^2}}$

= shortest distance from P to p.

important info to remember!!

Two lines in 3-space are either:

1. equal
2. parallel but not equal
3. intersect in one point
4. skew

note: In cases 2 and 3 there is a unique plane containing the two lines.

Two planes in 3-space are either:

1. equal
2. parallel but not equal
3. intersect in a line

note: In 3, the cross product of normal vectors for the planes is a direction vector for the line of intersection.

A line ℓ and a plane p satisfy one of:

1. ℓ is contained in p
2. ℓ is parallel to p but not contained in p
3. ℓ intersects p in one point

note: In 1 or 2, a direction vector for ℓ will be perpendicular to a normal vector for p.

<u>Example</u> The two lines

$$\ell_1 : \begin{cases} x = 5t - 3 \\ y = -t + 1 \\ z = -t + 1 \end{cases} \qquad \ell_2 : \begin{cases} x = 3t \\ y = 4t + 5 \\ z = -t \end{cases}$$

intersect in one point.

Find an equation for the plane p containing ℓ_1 and ℓ_2.

<u>Use previously described strategy!!</u>

(i) Find point of intersection of ℓ_1 and ℓ_2.

$(-3, 1, 1)$

$$\begin{cases} 5t - 3 = 3s \\ -t + 1 = 4s + 5 \\ -t + 1 = -s \end{cases}$$

(ii) Find $\vec{N}_p$. Take $\vec{N}_p = \vec{d}_{\ell_1} \times \vec{d}_{\ell_2}$

$$\vec{N}_p = \langle 5, -1, -1 \rangle \times \langle 3, 4, -1 \rangle$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 5 & -1 & -1 \\ 3 & 4 & -1 \end{vmatrix} = 5\vec{i} + 2\vec{j} + 23\vec{k}$$

$$p: \quad 5(x - (-3)) + 2(y - 1) + 23(z - 1) = 0$$

$$5x + 2y + 23z - 10 = 0$$

A curve C in xyz-space can be described by parametric equations:

$$C: \begin{cases} x = f(t) \\ y = g(t) \\ z = h(t) \end{cases} \qquad a \le t \le b$$

Think of these equations as representing the motion of an object in 3-space which is located at the point $(x(t), y(t), z(t))$ at time t.

It is convenient to express these equations in <u>vector form</u>.

$$C: \vec{r}(t) = \langle f(t), g(t), h(t) \rangle \quad , \quad a \le t \le b$$

This allows us to use vector operations and to take derivatives:

$$\vec{r}\,'(t) = \langle f'(t), g'(t), h'(t) \rangle$$

This derivative can be described as:

$$\vec{r}\,'(t_0) = \lim_{h \to 0} \frac{1}{h}\left(\vec{r}(t_0+h) - \vec{r}(t_0)\right).$$

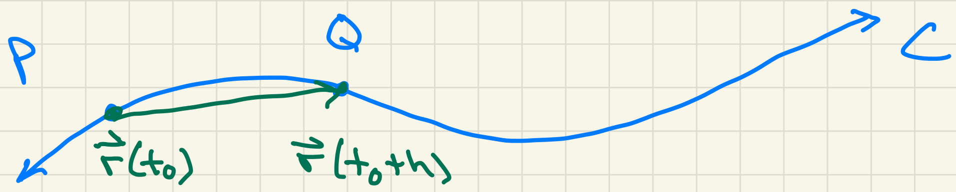
$=$ velocity at time t_0

Fact If we think of $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ describing the motion of an object along a curve C, then the velocity at time $t = t_0$ is $\vec{r}\,'(t_0)$, and this is a vector tangent to C at the point where $t = t_0$.

Also, the speed of object at time t_0 is

$$s(t) = |\vec{r}\,'(t_0)| = \sqrt{x'(t_0)^2 + y'(t_0)^2 + z'(t_0)^2}$$

Why?? The displacement of object over time interval between t_0 and t_0+h is $\vec{r}(t_0+h) - \vec{r}(t_0)$



the velocity over that interval is $\frac{1}{h}\left(\vec{r}(t_0+h) - \vec{r}(t_0)\right)$.

Taking the limit as $h \to 0$ gives the "instantaneous velocity" at time t_0 as $\vec{r}\,'(t_0)$.

Example The curve associated with the vector function

$$\vec{r}(t) = t\vec{d} + \vec{r}_0 = t\langle a,b,c\rangle + \langle x_0, y_0, z_0\rangle$$

is a straight line and

$$\vec{r}\,'(t) = \vec{d} = \langle a,b,c\rangle$$

So these equations represent the motion of an object in 3-space with constant velocity.

Note: Having constant velocity is a much stronger condition than having constant speed.

Having constant velocity guarantees the object is moving along a straight line.