Lines and Planes — RECAP

Lines in 3-space

Each line ℓ in xyz-space has a parametrization:

$$\ell : \begin{cases} x = at + x_0 \\ y = bt + y_0 \\ z = ct + z_0 \end{cases}$$

for some constants a, b, c and x_0, y_0, z_0. The vector $\vec{d}_\ell = \langle a, b, c \rangle$ is called a <u>direction vector</u> for ℓ (it is parallel to ℓ). The point on ℓ where $t = 0$ is (x_0, y_0, z_0).

or write this as:

$$Ax + By + Cz = -D$$

Planes in 3-space

Every plane p in 3-space has an equation of the form:

$$p : Ax + By + Cz + D = 0$$

(called a "linear equation with 3 variables".)

The vector $\vec{N} = \langle A, B, C \rangle$ is perpendicular to p, it is called a <u>normal vector</u> for p.

from webwork12

D. $z = 5$

9. **(1 point)** Library/UMN/calculusStewartET/s_12_5_34.pg

Find the linear equation of the plane through the point $(1,2,3)$ P and contains the line represented by the vector equation $\mathbf{r}(t) = \langle 3t, 5-3t, 1-4t \rangle$. ℓ

Equation: __________

10. **(1 point)** Library/Union/setMVlinesplanes/an12_6_17.pg

The line $L(t) = \langle 5-3t, 19-5t, 8-3t \rangle$ intersects the plane $x - 5y - 5z = -19$ at the point __________ when $t =$ __________.

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11. **(1 point)** Library/Union/setMVlinesplanes/an12_6_15.pg

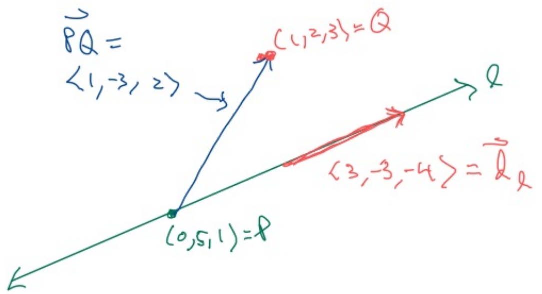
Consider the line $L(t) = \langle 3t-3, 5t-2, 5t-4 \rangle$. Then:

L is [?] to the plane $4.5x + 7.5y + 7.5z = 75$
L is [?] to the plane $35x - 19y - 2z = -87$
L is [?] to the plane $6x + 10y + 10z = 18$
L is [?] to the plane $5z - 4y = 14$

$$\ell: \begin{cases} x = 3t \\ y = -3t + 5 \\ z = -4t + 1 \end{cases}$$

$$\vec{d}_\ell = \langle 3, -3, -4 \rangle$$
$$P = (0, 5, 1)$$

picture:



The vectors $\vec{PQ}$ and $\vec{d}_\ell$ are non-parallel vectors in the plane p. So a normal vector for p is:

$$\vec{d}_\ell \times \vec{PQ} = \langle 3, -3, 4 \rangle \times \langle 1, -3, 2 \rangle$$
$$= -18\vec{i} - 10\vec{j} - 6\vec{k}$$

conclusion An equation for the plane is

$$-18(x-1) - 10(y-2) - 6(z-3) = 0$$

$$9x + 5y + 3z - 28 = 0$$

use this

Strategy for finding the equation of a plane p

(i) Find a point $P = (x_0, y_0, z_0)$ on p.

(ii) Find a vector $\vec{N}_p = \langle a, b, c \rangle$ perpendicular to p ("normal vector").

Then an equation for the plane is

$$p: A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

important info to remember !!

Two lines in 3-space are either:

1. equal
2. parallel but not equal
3. intersect in one point
4. skew

note: In cases 2 and 3 there is a unique plane containing the two lines.

Two planes in 3-space are either:

1. equal
2. parallel but not equal
3. intersect in a line

note: In 3, the cross product of normal vectors for the planes is a direction vector for the line of intersection.

A line ℓ and a plane p satisfy one of:

1. ℓ is contained in p
2. ℓ is parallel to p but not contained in p
3. ℓ intersects p in one point

note: In 1 or 2, a direction vector for ℓ will be perpendicular to a normal vector for p.

Example Consider the plane $p: 3x-2y-z+6=0$.

Does the line $\ell: x=t-1, y=3t+1, z=-3t+1$ intersect p?
If so, how many points of intersection?

First recall info on previous page.

$$\vec{d} = \vec{d}_\ell = \langle 1,3,-3\rangle$$

$$\vec{N} = \vec{N}_p = \langle 3,-2,-1\rangle$$

$$\vec{d}\cdot\vec{N} = (1)(3)+(3)(-2)+(-3)(-1) = 0$$

Conclude: the line ℓ is $\perp$ to $\vec{N}$,

So ℓ is either ① contained in p or ② parallel to p but not contained in p.

To check ①, plug the equations for ℓ into LHS of equation for p, and simplify:

$$3(t-1)-2(3t+1)-(-3t+1)+6$$
$$= 3t-3-6t-2+3t-1+6$$
$$= 0$$

$(t-1, 3t+1, -3t+1)$

conclude: Each point on the line ℓ is on the plane p so ① holds. There are infinitely many points of intersection.

note: In this problem we could directly started with this calculation

Problem If ℓ_1 and ℓ_2 are parallel lines and $\ell_1 \neq \ell_2$ what procedure could be used to find an equation for the plane p containing ℓ_1 and ℓ_2?

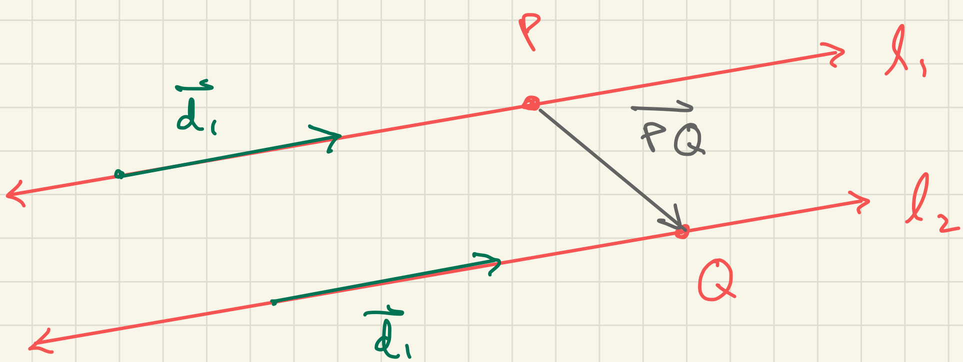
Answer

① Find a direction vector $\vec{d}_1$ for ℓ_1. (NOTE: $\vec{d}_1$ is also a direction vector for ℓ_2 because the 2 lines are parallel.)

② Find a point P on ℓ_1 and a point Q on ℓ_2.

③ The vector $\vec{N} = \vec{PQ} \times \vec{d}_1$ is a normal vector for p (because $\vec{PQ}$ and $\vec{d}_1$ are non-parallel vectors in p).

④ Use $\vec{N}$ and P to find an equation for ℓ.



Note: In step ④ it would be legitimate to use $\vec{N}$ and Q to find an equation for p. If you do you will end up with the same equation for p.

Comment about lines in 3-space

We have highlighted using parametric equations to describe a line in 3-space, but it is also common to describe a line as the intersection of two planes.

example The two lines in this webwork13 problem:

ℓ_1

15. **(1 point)** `Library/Michigan/Chap12Sec4/Q11.pg`

Find an equation for the plane containing the line in the xy-plane where $y = 3$, and the line in the xz-plane where $z = 4$.

equation: ____________________

ℓ_2

have been described as intersections of planes:

ℓ_1 = intersection of $z=0$ and $y=3$.

ℓ_2 = intersection of $y=0$ and $z=4$.

which have parametric equations in vector form:

$$\ell_1: \quad \vec{r}_1(t) = \langle t, 3, 0 \rangle$$

$$\ell_2: \quad \vec{r}_2(t) = \langle t, 0, 4 \rangle$$

These two lines have a common direction vector $\vec{d}_1 = \langle 1, 0, 0 \rangle$.

Now use procedure on previous page to solve #15 . . . .

<u>another example</u> The y-axis in xyz-space consists of all points (x, y, z) where x and z both equal 0. In other words, the y-axis is the intersection of the planes $x=0$ (yz-plane) and $z=0$ (xy-plane).

A parametric description of the y-axis might be $\vec{r}(t) = \langle 0, t, 0 \rangle$.

In scalar form this is

$$\text{y-axis} \begin{cases} x = 0 & = 0t + 0 \\ y = t & = 1t + 0 \\ z = 0 & = 0t + 0 \end{cases}$$

↑ direction vector $\langle 0, 1, 0 \rangle = \vec{j}$
point $P = (0,0,0)$

now back to Chapter 13 . . . .

A curve C in xyz-space can be described by parametric equations:

may or may not have a restriction like this.

$$C: \begin{cases} x = f(t) \\ y = g(t) \\ z = h(t) \end{cases} \qquad a \le t \le b$$

Or express these equations in vector form:

$$C: \vec{r}(t) = \langle f(t), g(t), h(t) \rangle \quad , \; a \le t \le b$$

we say $\vec{r}(t)$ is a vector function

We can then differentiate:

$$\vec{r}\,'(t) = \langle f'(t), g'(t), h'(t) \rangle$$
$$= \lim_{h \to 0} \frac{1}{h} \left(\vec{r}(t_0 + h) - \vec{r}(t_0) \right).$$

In either form we often interpret these equations as representing the motion of an object in 3-space, which is located at the point $(x(t), y(t), z(t))$ at time t.

then $\vec{r}\,'(t_0) = \vec{v}(t)$ represents the velocity vector of the object at time t_0. And this vector $\vec{r}\,'(t_0)$ is tangent to the curve C at the point where $t = t_0$.

Also $s(t_0) = |\vec{r}\,'(t_0)| = \sqrt{x'(t_0)^2 + y'(t_0)^2 + z'(t_0)^2}$

$= \text{speed at time } t_0$

$\vec{a}(t_0) = \vec{r}\,''(t_0) = \text{acceleration vector at time } t_0.$

Example $\quad C: \begin{cases} x = 4t-1 \\ y = t^2 - t - 1 \\ z = 2t^3 - 3t^2 + 5 \end{cases}$

In vector form this would be

$$C: \vec{r}(t) = \langle 4t-1, t^2-t-1, 2t^3-3t^2+5 \rangle$$

Find equations for the line ℓ tangent to C at $(3,-1,4)$.

At $(3,-1,4)$, $t=1$.

$\vec{r}\,'(t) = \langle 4, 2t-1, 6t^2-6t \rangle$

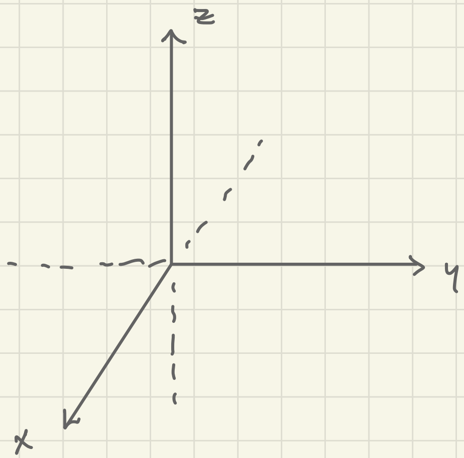
$\vec{r}\,'(1) = \langle 4, 1, 0 \rangle = \vec{d}_\ell$

So ℓ has direction vector $\vec{d}_\ell = \langle 4, 1, 0 \rangle$ and goes through the point $(3,-1,4)$, and

$$\ell: \begin{cases} x = 4t + 3 \\ y = t - 1 \\ z = 4 \end{cases} \quad \leftarrow z = 0 \cdot t + 4 = 4$$

Example $\quad C: \begin{cases} x = 4t-1 \\ y = t^2 - t - 1 \\ z = 2t^3 - 3t^2 + 5 \end{cases}$

Any top-most points on C? right-most? front-most?
ⓐ ⓑ ⓒ



← Our standard positioning of axes:

x = front / back

y = right / left

z = top / bottom

ⓐ $z(t) = 2t^3 - 3t^2 + 5$ ← measures the vertical position of object

$\Rightarrow$ $z'(t) > 0$ means object moving up at time t,
$z'(t) < 0$ means object moving down at time t

here: $z'(t) = 6t^2 - 6t = 6t(t-1)$

$z' > 0$ | $z' < 0$ | $z' > 0$ → t-axis
up | 0 | down | 1 | up

$\Rightarrow$ topmost point when $t = 0$, $(x,y,z) = (-1,-1,5)$

ⓑ $y'(t) = 2t - 1$

$y' < 0$ | $y' > 0$
left | $1/2$ | right

$\Rightarrow$ no right-most point

ⓒ $x'(t) = 4$ $\Rightarrow$ no front-most point.

(always moving frontward)