Vector Functions and Curves in 3-space

A curve C in xyz-space is described by a parametrization

scalar form:

$$C: \begin{cases} x = f(t) \\ y = g(t) \\ z = h(t) \end{cases}$$

or in vector form:

$$C: \vec{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$$

($\vec{r}(t)$ is called a 'vector function'.)

In either form the curve C consists of all points $(f(t), g(t), h(t))$ for all possible values of the parameter t.

It is useful to think of the equations as describing the motion of an object in 3-space which traces out the curve C. (At time t the object is at $(f(t), g(t), h(t))$.)

Then

$$\vec{r}\,'(t) = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle = \langle f'(t), g'(t), h'(t) \rangle$$

$=$ velocity (vector) of object at time t

The vector $\vec{r}\,'(t_0)$ is a direction vector for the line tangent to the curve C at the point P where $t = t_0$.

Also the speed of the object at time t is

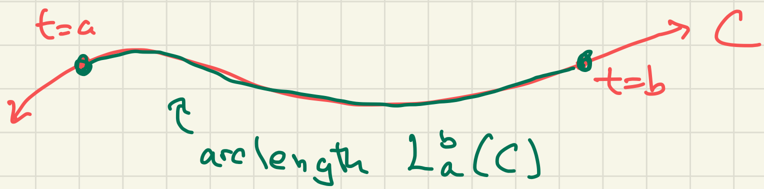
$$\text{speed}(t) = |\vec{r}\,'(t)|$$

By integrating the speed we can determine how far the object has travelled over a time interval $[a,b]$

$$\text{Distance travelled from } t=a \text{ to } t=b = \int_a^b |\vec{r}\,'(t)|\,dt$$

If the object does not retrace a portion of C from time $t=a$ to time $t=b$ then this determines 'arclength'

$$\boxed{L_a^b(C) = \int_a^b |\vec{r}\,'(t)|\,dt}$$



Examples

(1) $C: \vec{r}(t) = \langle 3,-2,-1\rangle t + \langle -1,0,7\rangle = \langle 3t-1,-2t,-t+7\rangle$

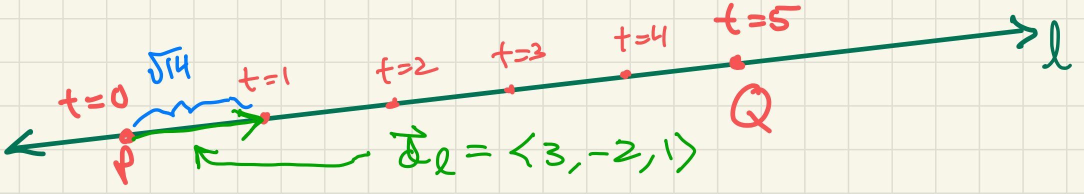
$\Rightarrow \quad \vec{r}\,'(t) = \langle 3,-2,-1\rangle$ ← Object has constant velocity!

$\text{speed}(t) = (3^2+(-2)^2+(-1)^2)^{1/2} = \sqrt{14}$

The arclength of C between $P=(-1,0,7)$ (where $t=0$) to $Q=(14,-10,2)$ (where $t=5$) is

$$L(P,Q) = \int_0^5 \sqrt{14}\,dt = \sqrt{14}\,t\Big|_0^5 = 5\sqrt{14}$$

Interpretation This curve C is a line ℓ with direction vector $\langle 3,-2,1\rangle$



(2) $C: \vec{r}(t) = \langle \cos t, \sin t, 3t \rangle$ helix!

$\vec{r}\,'(t) = \langle -\sin t, \cos t, 3 \rangle$

object has constant speed but not constant velocity.

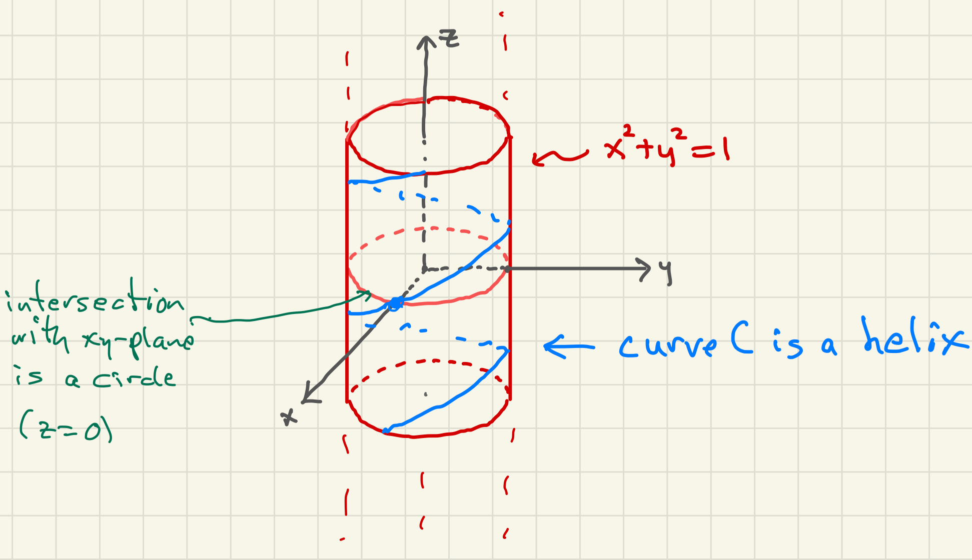
$\text{speed}(t) = |\vec{r}\,'(t)| = \left((-\sin t)^2 + (\cos t)^2 + 3^2 \right)^{1/2} = \sqrt{10}$

length of C from $P = (1,0,0)$ (where $t=0$) to $Q = (-1, 0, 9\pi)$ (where $t = 3\pi$) is

$$L_P^Q(C) = \int_0^{3\pi} \sqrt{10}\, dt = 3\sqrt{10}\,\pi$$

from 11-19 class notes:

Note: Each point $(\cos t, \sin t, 3t)$ on C satisfies the equation $x^2+y^2=1$. The set of all points satisfying $x^2+y^2=1$ is a cylinder in 3-space whose axis is the z-axis.



(3) $C: \vec{r}(t) = \langle t\cos t, t\sin t, 3t \rangle$

Find the arclength of C from origin $O=(0,0,0)$ (where $t=0$) to $Q=(-3\pi, 0, 9\pi)$ (where $t=3\pi$).

Use the product rule to determine velocity:

$$\vec{r}\,'(t) = \langle \cos t - t\sin t, \sin t + t\cos t, 3 \rangle$$

and speed:

$$|\vec{r}\,'(t)| = \left((\cos t - t\sin t)^2 + (\sin t + t\cos t)^2 + 3^2 \right)^{1/2}$$
$$= \left(\cos^2 t + t^2\sin^2 t + \sin^2 t + t^2\cos^2 t + 9 \right)^{1/2}$$
$$= \sqrt{10+t^2}$$

So arclength equals

$$L_O^Q(C) = \int_0^{3\pi} \sqrt{10+t^2}\, dt$$

$$= \int_{t=0}^{3\pi} 10\sec^3\theta \, d\theta$$

trig substitution:
$t = \sqrt{10}\tan\theta$
$10+t^2 = 10\sec^2\theta$
$dt = \sqrt{10}\sec^2\theta$

integration by parts, or table look-up.

$$= 10 \cdot \tfrac{1}{2}\left(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right)\Big|_{t=0}^{3\pi}$$

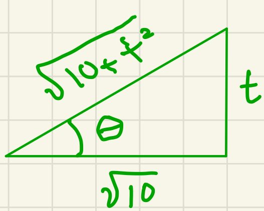
$$= 5\left(\frac{\sqrt{10+9\pi^2}}{\sqrt{10}} \cdot \frac{t}{\sqrt{10}} + \ln\left| \frac{\sqrt{10+t^2}}{\sqrt{10}} + \frac{t}{\sqrt{10}} \right| \right)\Big|_{t=0}^{3\pi}$$

$$= 5\left(\frac{3\pi\sqrt{10+9\pi^2}}{10} + \ln\left(\frac{\sqrt{10+9\pi^2} + 3\pi}{\sqrt{10}} \right) \right)$$

right triangle analysis

$\sec\theta = \dfrac{\text{hyp}}{\text{adj}} = \dfrac{\sqrt{10+t^2}}{\sqrt{10}}$

$\tan\theta = t/\sqrt{10}$



more on (3): $C: \vec{r}(t) = \langle t\cos t, t\sin t, 3t \rangle$

Any point on C has the form $(x,y,z) = (t\cos t, t\sin t, 3t)$. So

$x^2+y^2 = t^2\cos^2 t + t^2 \sin^2 t = t^2(\sin^2 t + \cos^2 t) = t^2 = \left(\frac{z}{3}\right)^2 = z^2/9$

This shows that the curve C lies on the surface with equation $9x^2+9y^2-z^2=0$. What does this surface look like? —Will return to this question in a later class.

$\boxed{\text{its a cone}}$