A power series centered at a has form

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

It defines a function $f(x)$ whose domain is an interval I with endpoints $a-R$ and $a+R$, where $R \geq 0$ is the radius of convergence of the power series.

R is called the "radius of convergence" of the power series, and I is its "interval of convergence".

Strategy for determining interval of convergence:

(i) Use Ratio Test to find R.

(ii) Use other tests to determine convergence at the two endpoints: $x = a-R$, $x = a+R$

(iii) In the cases where $R=0$ or $R=\infty$, step (2) will be unnecessary.

Question: Which functions $f(x)$ can be expressed as

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \quad \text{where } R>0 \text{ ?}$$

from Stewart

Geometric Series

Table 1 Important Maclaurin Series and Their Radii of Convergence

	$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$	$I = (-1, 1)$	$R = 1$
✓	$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$	$I = (-\infty, \infty) = \mathbb{R}$	$R = \infty$
	$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$		$R = \infty$
	$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$		$R = \infty$
$\arctan(x) =$ ✓	$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$		$R = 1$
✓	$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$		$R = 1$
"binomial series"	$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$		$R = 1$

terminology:

A "Maclaurin series" is a power series centered at $a=0$.

We will not discuss the binomial series (b/c its a bit technical) but will examine the others.

Example For $f(x) = \ln(1+x)$, $x > -1$:

Geometric Series for $-x$

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n$$

Integrate this to get

$$\ln(1+x) = \int \frac{1}{1+x}\,dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$$

Since $f(0) = \ln(1+0) = 0$, C equals 0 and

$$\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$$

The "Taylor series at $x=a$" for a function $f(x)$ is the power series

$$T_{x=a}(f) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

(recall: $f^{(n)}(x) = n^{th}$ derivative of $f(x)$ and
$f^{(n)}(a) = f^{(n)}(x)$ evaluated at the number $x=a$.)

Fact: If $f(x)$ equals a power series centered at $x=a$ with $R>0$ then that series must be the Taylor series $T_{x=a}(f)$.

There is a result called "Taylor's Theorem" (Stewart, page 802) which gives a condition that guarantees that a given function will equal its Taylor Series. (This is an expanded version of the Mean Value Theorem from Calculus 1.) One application:

Fact (*): If $|f^{(n)}(x)| \leq n!$ for all $n \geq N$ and for all x in an open interval $(a-d, a+d)$ for some numbers $N>0$ and $d>0$ then $f(x)$ equals its Taylor series $T_{x=a}(f)$.

Example $f(x) = 5x^2 - x + 3$

Find $T_{x=3}(f)$. $a=3$

$f^{(0)}(x) = 5x^2 - x + 3$ $\qquad f^{(0)}(3) = 45$

$f^{(1)}(x) = 10x - 1$ $\qquad f^{(1)}(3) = 29$

$f^{(2)}(x) = 10$ $\qquad f^{(2)}(3) = 10$

$f^{(n)}(x) = 0$ for $n \geq 3$. $\qquad f^{(n)}(3) = 0$

$T_{x=3}(f) =$

$$= \frac{45}{0!}(x-3)^0 + \frac{29}{1!}(x-3)^1 + \frac{10}{2!}(x-3)^2 + \frac{0}{3!}(x-3)^3 + \cdots$$

$$= 45 + 29(x-3) + 5(x-3)^2$$

$$= f(x)$$

check? $45 + 29(x-3) + 5(x^2 - 6x + 9)$

$$= 45 + 29x - 87 + 5x^2 - 30x + 45$$

$$= 5x^2 + (29-30)x + (45 - 87 + 45)$$

$$= 5x^2 - x + 3$$

Example $f(x) = \cos(x)$, $a=0$

① What is $f^{(n)}(x)$?
② What is $f^{(n)}(0)$?
③ What is $T_{x=0}(f)$ for $f(x)=\cos(x)$?

For ① and ②:

$$\begin{aligned} f^{(0)}(x) &= \cos x & f^{(0)}(0) &= 1 \\ f^{(1)}(x) &= -\sin x & f^{(1)}(0) &= 0 \\ f^{(2)}(x) &= -\cos x & f^{(2)}(0) &= -1 \\ f^{(3)}(x) &= \sin x & f^{(3)}(0) &= 0 \\ f^{(4)}(x) &= \cos x & f^{(4)}(0) &= 1 \\ &\vdots & &\vdots \end{aligned}$$

repeats in blocks of four.

Summary $f^{(2n)}(0) = (-1)^n$, $f^{(2n+1)}(0) = 0$

$\Rightarrow$ all odd powers of x^n drop out in $T_{x=0}(f)$.

For ③

$$T_{x=0}(f) = \frac{1}{0!}x^0 + \frac{0}{1!}x^1 + \frac{-1}{2!}x^2 + \frac{0}{3!}x^3 + \frac{1}{4!}x^4 + \cdots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

Note that, for all x, $|f^{(n)}(x)| \leq 1$ (b/c $f^{(n)}(x) = \pm\sin x$ or $\pm\cos x$)
so Fact ⊛ shows that $\cos(x) = T_{x=0}(f)$

Result: $\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} x^{2n}$$

(as indicated in Stewart's table.)

Example $\int \cos(x^2)\,dx = ??$

note: This integral cannot be worked in closed form using elementary functions, (see discussion in Stewart — copied below) but it can be worked out using power series:

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

$$\Rightarrow \cos(x^2) = 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \frac{x^{16}}{8!} - \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n}}{(2n)!}$$

$$\Rightarrow \int \cos(x^2)\,dx = x - \frac{x^5}{5 \cdot 2!} + \frac{x^9}{9 \cdot 4!} - \frac{x^{13}}{13 \cdot 6!} + \frac{x^{17}}{17 \cdot 8!} - \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+1}}{(4n+1)(2n)!} + C$$

from Stewart:

The functions that we have been dealing with in this book are called **elementary functions**. These are the polynomials, rational functions, power functions (x^n), exponential functions (b^x), logarithmic functions, trigonometric and inverse trigonometric functions, hyperbolic and inverse hyperbolic functions, and all functions that can be obtained from these by the five operations of addition, subtraction, multiplication, division, and composition. For instance, the function

$$f(x) = \sqrt{\frac{x^2 - 1}{x^3 + 2x - 1}} + \ln(\cosh x) - xe^{\sin 2x}$$

is an elementary function.

If f is an elementary function, then f' is an elementary function but $\int f(x)\,dx$ need not be an elementary function. Consider $f(x) = e^{x^2}$. Since f is continuous, its integral exists, and if we define the function F by

$$F(x) = \int_0^x e^{t^2}\,dt$$

then we know from Part 1 of the Fundamental Theorem of Calculus that

$$F'(x) = e^{x^2}$$

Thus $f(x) = e^{x^2}$ has an antiderivative F, but it has been proved that F is not an elementary function. This means that no matter how hard we try, we will never succeed in evaluating $\int e^{x^2}\,dx$ in terms of the functions we know. (In Chapter 11, however, we will see how to express $\int e^{x^2}\,dx$ as an infinite series.) The same can be said of the following integrals:

$$\int \frac{e^x}{x}\,dx \qquad \int \sin(x^2)\,dx \qquad \int \cos(e^x)\,dx$$

$$\int \sqrt{x^3 + 1}\,dx \qquad \int \frac{1}{\ln x}\,dx \qquad \int \frac{\sin x}{x}\,dx$$

In fact, the majority of elementary functions don't have elementary antiderivatives. You may be assured, though, that the integrals in the following exercises are all elementary functions.

and its close relative $\int \cos(x^2)\,dx$.