Now back to Chapter 12 :

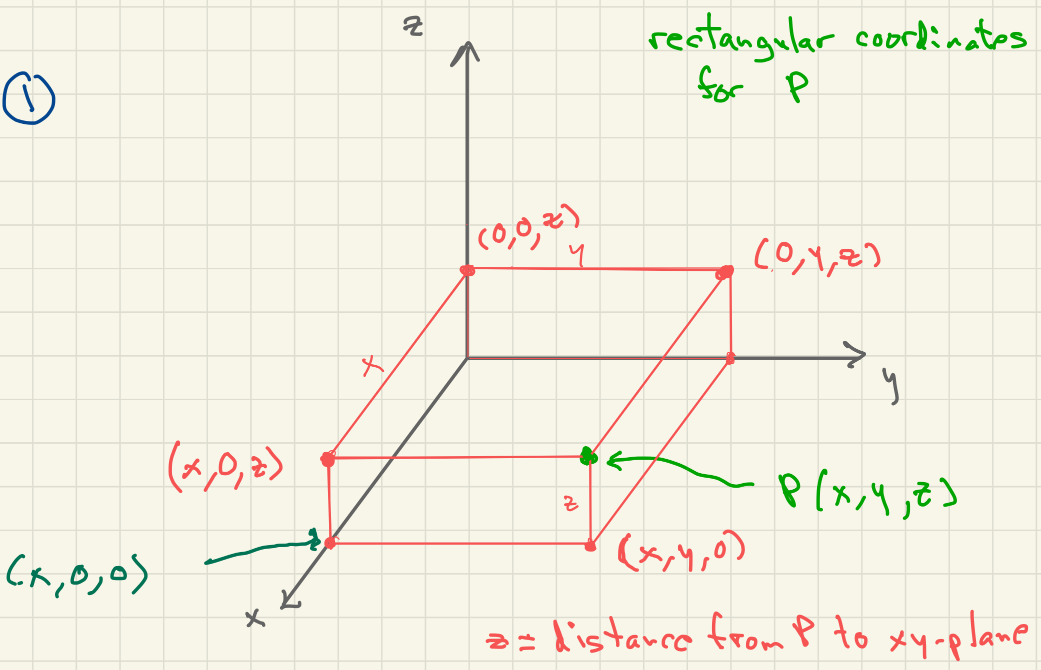
Immediate Goal (Section 12.5)

Find equations to describe lines and planes in $\mathbb{R}^3$.

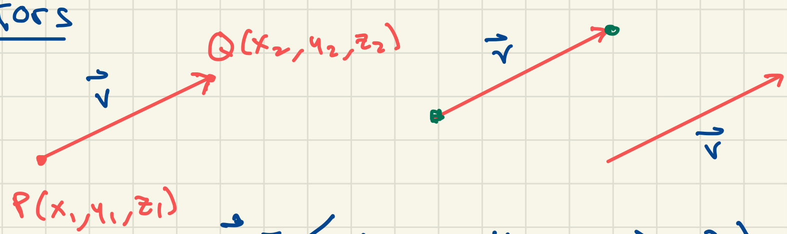
First, Recall !

1. rectangular coordinates in $\mathbb{R}^3$
2. vectors
3. Dot product of vectors
4. cross product of vectors

(1)



② Vectors



$$\vec{v} = \langle x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1 \rangle$$
$$= (x_2 - x_1)\vec{i} + (y_2 - y_1)\vec{j} + (z_2 - z_1)\vec{k}$$

③ Dot Product

$\vec{a} = \langle a_1, a_2, a_3 \rangle \quad \vec{b} = \langle b_1, b_2, b_3 \rangle$

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 = \text{a scalar}$$

- $\vec{a} \cdot \vec{a} = |\vec{a}|^2$
- $\cos\theta = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}|\,|\vec{b}|}$, θ = angle between $\vec{a}$ and $\vec{b}$
- $\vec{a} \perp \vec{b} \iff \vec{a} \cdot \vec{b} = 0$

④ Cross Product

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \text{a vector}$$

- $\vec{a} \times \vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$.
- $|\vec{a} \times \vec{b}| = |\vec{a}|\,|\vec{b}|\,|\sin\theta|$
- $(\vec{a}, \vec{b}, \vec{a} \times \vec{b})$ satisfies right-hand rule.

Example For constant a, let

$$\vec{u} = \langle 2, 6, -2 \rangle \quad \text{and} \quad \vec{v} = \langle 2a+1, 3a, a^2 \rangle$$

① Find any values of a for which $\vec{u}$ is parallel to $\vec{v}$.

$\boxed{\text{To be parallel means there is a scalar } k \text{ with } k\vec{u} = \vec{v}:}$

here

$$\begin{cases} 2k = 2a+1 \\ 6k = 3a \\ -2k = a^2 \end{cases} \Rightarrow \cdots \Rightarrow \quad a = -1 \ (\text{and } k = -1/2)$$

Can you fill in details?

② Find any values of a for which $\vec{u}$ is perpendicular to $\vec{v}$.

$$\vec{u} \cdot \vec{v} = 2(2a+1) + 6(3a) - 2a^2 = -2a^2 + 22a + 2$$
$$= -2(a^2 - 11a - 1)$$

When does $a^2 - 11a - 1 = 0$?

$$a = \frac{11 \pm \sqrt{11^2 + 4}}{2} = \frac{11 \pm 5\sqrt{5}}{2}$$

Review
Class Notes from September 20:

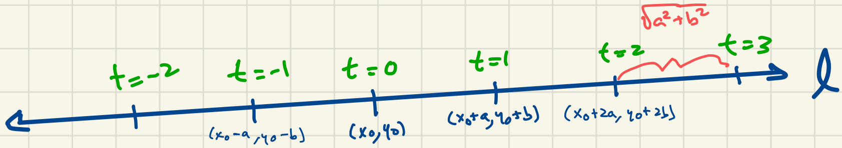
② Lines. If $a \neq 0, b, x_0, y_0$ are constants then

$$\ell : \begin{cases} x = at + x_0 \\ y = bt + y_0 \end{cases}$$

describes a nonvertical line ℓ in $\mathbb{R}^2$. This line goes thru (x_0, y_0) (when $t=0$) and (x_0+a, y_0+b) (when $t=1$) and has slope b/a.

A rectangular equation is $y - y_0 = \frac{b}{a}(x - x_0)$, which can also be written with the form $y = \frac{b}{a}x + \frac{ay_0 - bx_0}{a}$ or by $ay - bx = ay_0 - bx_0$.

The speed of this motion is $s(t) = \sqrt{(dx/dt)^2 + (dy/dt)^2} = \sqrt{a^2+b^2}$ so the object is moving with constant speed.



So this parametrization amounts to identifying ℓ as a number line in which the unit length is $\sqrt{a^2+b^2}$.

Note Well! The same line ℓ can be identified with many, many different number lines by rechoosing the unit length and/or rechoosing the "origin" (point wher $t=0$). So one line ℓ has many different parametrizations of the form described here. On top of that there are many parametric descriptions of ℓ that don't have constant speed.

In section 12.5, we will examine these parametrizations of lines more thoroughly.

Lines in 3-space

Each line ℓ in xyz-space has a parametrization:

$$\ell : \begin{cases} x = at + x_0 \\ y = bt + y_0 \\ z = ct + z_0 \end{cases}$$

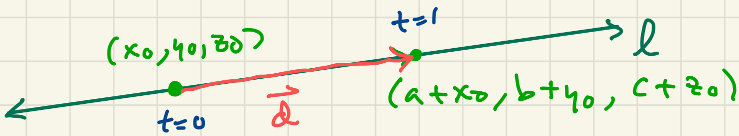
for some constants a, b, c and x_0, y_0, z_0.

Some comments

① A point on ℓ has coordinates $(x,y,z) = (at + x_0, bt + y_0, ct + z_0)$ for some number t.

② (x_0, y_0, z_0) is the point on ℓ having $t = 0$.

③ the point on ℓ with $t = 1$ is $(a + x_0, b + y_0, c + z_0)$.



④ The vector $\vec{d}$ from (x_0, y_0, z_0) to $(a + x_0, b + y_0, c + z_0)$ is $\vec{d} = \langle a, b, c \rangle$. It is called a <u>direction vector</u> for ℓ.

⑤ Observe in the parametrization

$$\ell : \begin{cases} x = at + x_0 \\ y = bt + y_0 \\ z = ct + z_0 \end{cases}$$

direction vector $\vec{d}$

point on ℓ where $t = 0$.

⑥ If we know a point (x_0, y_0, z_0) on ℓ and a direction vector $\vec{d} = \langle a, b, c \rangle$ then we can give a parametrization for ℓ.

example

$$\ell: \begin{cases} x = -2t + 1 & = at + x_0 \\ y = 3t - 2 & = bt + y_0 \\ z = t + 5 & = ct + z_0 \end{cases}$$

here $\vec{d} = \langle -2, 3, 1 \rangle$

$P = (1, -2, 5)$

① Is $(1,2,3)$ a point on ℓ? No!

$$\begin{cases} 1 = -2t + 1 \\ 2 = 3t - 2 \\ 3 = t + 5 \end{cases}$$

no solution

② Is $(3,-5,4)$ a point on ℓ? Yes

$$\left. \begin{array}{l} 3 = -2t + 1 \\ -5 = 3t - 2 \\ 4 = t + 5 \end{array} \right\}$$

$t = -1$ is a solution

③ Is ℓ parallel to the line ℓ_1?

$$\ell_1: \begin{cases} x = 6t = 1 \\ y = -9t = -2 \\ z = -3t = 5 \end{cases}$$

$\vec{d} = \langle -2, 3, 1 \rangle$

$\vec{d}_1 = \langle 6, -9, -3 \rangle$

parallel? Yes

b/c $\vec{d}_1 = -3\vec{d}$

④ Does $\ell = \ell_1$? No because $P = (1,-2,5)$ is on ℓ but not on ℓ_1.

Alternate Notation

The equations

$$\ell : \begin{cases} x = at + x_0 \\ y = bt + y_0 \\ z = ct + z_0 \end{cases}$$

are called parametric equations for ℓ.

We can also write them as a vector equation for ℓ

$$\ell : \vec{r}(t) = t\vec{d} + \vec{r}_0$$

where

$$\vec{r}(t) = \langle x, y, z \rangle = \langle x(t), y(t), z(t) \rangle$$

$\vec{d} = \langle a, b, c \rangle$ = direction vector for ℓ

$\vec{r}_0 = \langle x_0, y_0, z_0 \rangle$ = point on ℓ

example

Find equations for the line containing points

$P = (-1, 3, 4)$ and $Q = (2, 2, -1)$.

$\boxed{\text{Strategy: First identify (i) a point on the line, and (ii) a direction vector for the line.}}$

Always use this strategy!

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