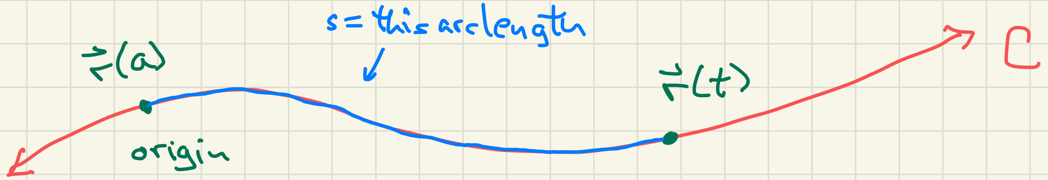
Geometry of curves in 3-space (section 13.3)

Every curve C in xyz-space has a parametrization:

$$C: \vec{r} = \vec{r}(t) = \langle f(t), g(t), h(t) \rangle$$

which describes the motion of an object that traces out the curve C. Of course there are many different ways that an object can move and trace out the same curve C. So, in order to focus on the geometry of the curve C rather than attributes of a motion, we will introduce a new variable s, and re-imagine $\vec{r} = \vec{r}(t)$ as a function of s.



First choose a time value $t = a$ and consider the corresponding point on C to be the "origin point." Then define $s = s(t)$ to be the arclength of C between $\vec{r}(a)$ and $\vec{r}(t)$.

By the arclength formula,

$$s = \int_a^t |\vec{r}\,'(u)|\, du$$

and the Fundamental Theorem of Calculus shows that

$$\frac{ds}{dt} = |\vec{r}\,'(t)|$$

Typically we will not be interested in determining s (which would require working an integral) but will instead focus on using the formula

$$\boxed{\frac{ds}{dt} = |\vec{r}\,'(t)|}$$

which says that s is an antiderivative for the speed.

When $\vec{r} = \vec{r}(t)$ is re-imagined as a function of s then we are essentially replacing the original motion along C with a new motion along C where the speed is $\left|\frac{d\vec{r}}{ds}\right|$ (rather than $\left|\frac{d\vec{r}}{dt}\right| = |\vec{r}\,'(t)|$). Since

$$\left|\frac{d\vec{r}}{ds}\right| = \left|\frac{d\vec{r}/dt}{ds/dt}\right| = \frac{|\vec{r}\,'(t)|}{ds/dt} = \frac{|\vec{r}\,'(t)|}{|\vec{r}\,'(t)|} = 1 \quad *$$

this new motion of an object along C has constant speed equal to 1. Since there is only one such motion on C, viewing $\vec{r}$ as a function of s will provide direct information about the geometry of C, rather than information about a motion along C.

---

* This equation actually does not make sense for a value of t where $\vec{r}\,'(t) = \vec{0}$. Remember that points on C where $\vec{r}\,'(t) = \vec{0}$ are "cusp" points and need to be examined separately. For this discussion we will assume that there no values of t where $\vec{r}\,'(t) = \vec{0}$.

Viewing $\vec{r}$ as a function of s leads to the following definitions useful for describing the geometry of a curve C.

- $\vec{T} = \vec{T}(t)$ = "unit tangent vector" defined by

$$\boxed{\vec{T} = \frac{d\vec{r}}{ds} = \frac{d\vec{r}/dt}{ds/dt} = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}}$$

- $\vec{N} = \vec{N}(t)$ = "unit normal vector" defined by

$$\boxed{\vec{N} = \frac{\vec{T}\,'(t)}{|\vec{T}\,'(t)|}}$$

Fact The vectors $\vec{T}$ and $\vec{N}$ are perpendicular.
(To show this: $0 = \frac{d}{dt}[1] = \frac{d}{dt}[\vec{T}\cdot\vec{T}] = 2\vec{T}(t)\cdot\vec{T}\,'(t)$.)

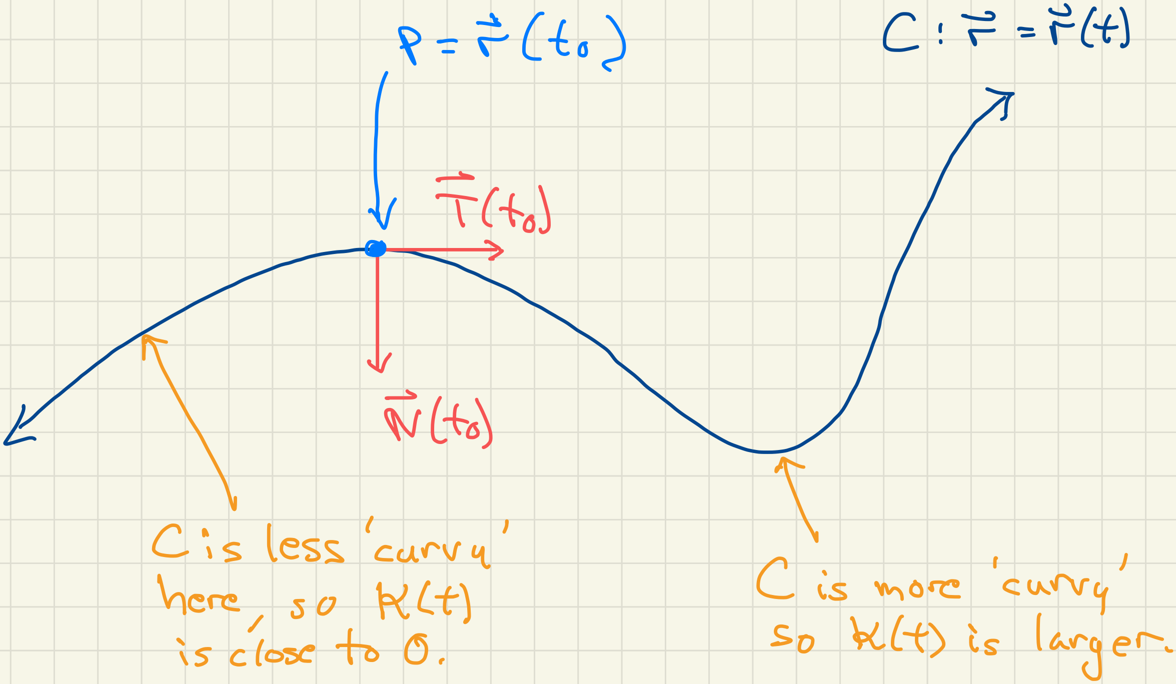
- If P is the point on C with time value t, then the plane through P which is parallel to the vectors $\vec{T}(t)$ and $\vec{N}(t)$ is called the "osculating plane" at P. It is plane through P that most closely contains a portion of the curve C near P. (A normal vector for this plane is $\vec{T}(t) \times \vec{N}(t)$.

- $\kappa = \kappa(t)$ = "the curvature function" for C, defined by

$$\boxed{\kappa(t) = \left|\frac{d\vec{T}}{ds}\right| = \left|\frac{d\vec{T}/dt}{ds/dt}\right| = \frac{|\vec{T}\,'(t)|}{|\vec{r}\,'(t)|}}$$

$\kappa(t)$ measures the 'curviness' of C at the point with time value t.

picture:



note: "less curvy" means "flatter"!

An extreme case would be if C is a line. In which case the curvature $\kappa(t)$ can be shown to equal 0 for every value of t.

special example $C: \vec{r}(t) = \langle R\cos t, R\sin t, 0\rangle$, $R>0$

C = circle of radius R in xy-plane. $\boxed{\text{Calculate } \kappa(t).}$

$$\vec{r}\,' = \langle -R\sin t, R\cos t, 0\rangle$$

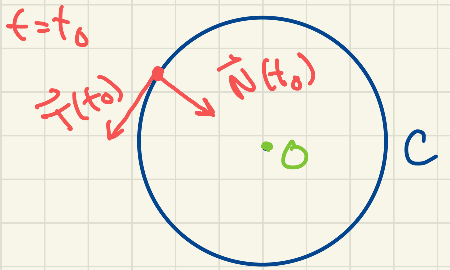
$$|\vec{r}\,'| = \left((-R\sin t)^2 + (R\cos t)^2 + 0^2\right)^{1/2} = R$$

$$\vec{T}(t) = \langle -\sin t, \cos t, 0\rangle$$

$$\vec{T}\,'(t) = \langle -\cos t, \sin t, 0\rangle = \vec{N}(t) \quad (\text{because } |\vec{T}\,'(t)| = 1)$$

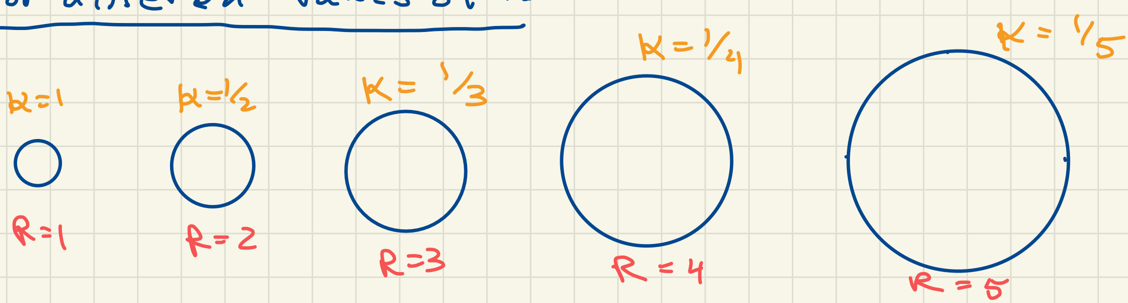
$$\kappa(t) = \frac{|T'(t)|}{|\vec{r}\,'(t)|} = \frac{1}{R}$$

picture:



← The vector $\vec{N}(t_0)$ points towards the center O of the circle.

For different values of R:



more curvy (larger κ) ⟷ less curvy (smaller κ)

For generic curve $C: \vec{r} = \vec{r}(t)$, $P = \vec{r}(t_0)$:

- $\kappa(t_0)$ measures the "curviness" of C at P.

- The plane $\mathcal{P}$ containing P and having normal vector $\vec{T}(t_0) \times \vec{N}(t_0)$ is plane which comes closest to containing C near P. (called osculating plane)

- The circle thru P which most closely approximates C near P is contained in the osculating plane at P. It has radius $1/\kappa(t_0)$ and its center is on the line through P with direction vector $\vec{N}(t_0)$: