Geometry of a Curve C — Key Constructs

$C: \vec{r} = \vec{r}(t) = \langle f(t), g(t), h(t) \rangle$

$\vec{r}\,'(t) = \langle f'(t), g'(t), h'(t) \rangle$ = velocity

$|\vec{r}\,'(t)| = \left(f'(t)^2 + g'(t)^2 + h'(t)^2 \right)^{1/2} = v(t)$ = speed

$\vec{T}(t) = \dfrac{1}{|\vec{r}\,'(t)|} \vec{r}\,'(t)$ = unit tangent vector

$|\vec{T}\,'(t)|$

$\kappa(t) = \dfrac{|\vec{T}\,'(t)|}{|\vec{r}\,'(t)|}$ = curvature

$\vec{N}(t) = \dfrac{1}{|\vec{T}\,'(t)|} \vec{T}\,'(t)$ = unit normal vector

$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$ = unit bi-normal vector

CAUTION: Must watch out for cusp points!

1. Both $\vec{T}(t_0)$ and $\kappa(t_0)$ are only defined for values $t = t_0$ where $\vec{r}\,'(t_0) \neq \vec{0}$.

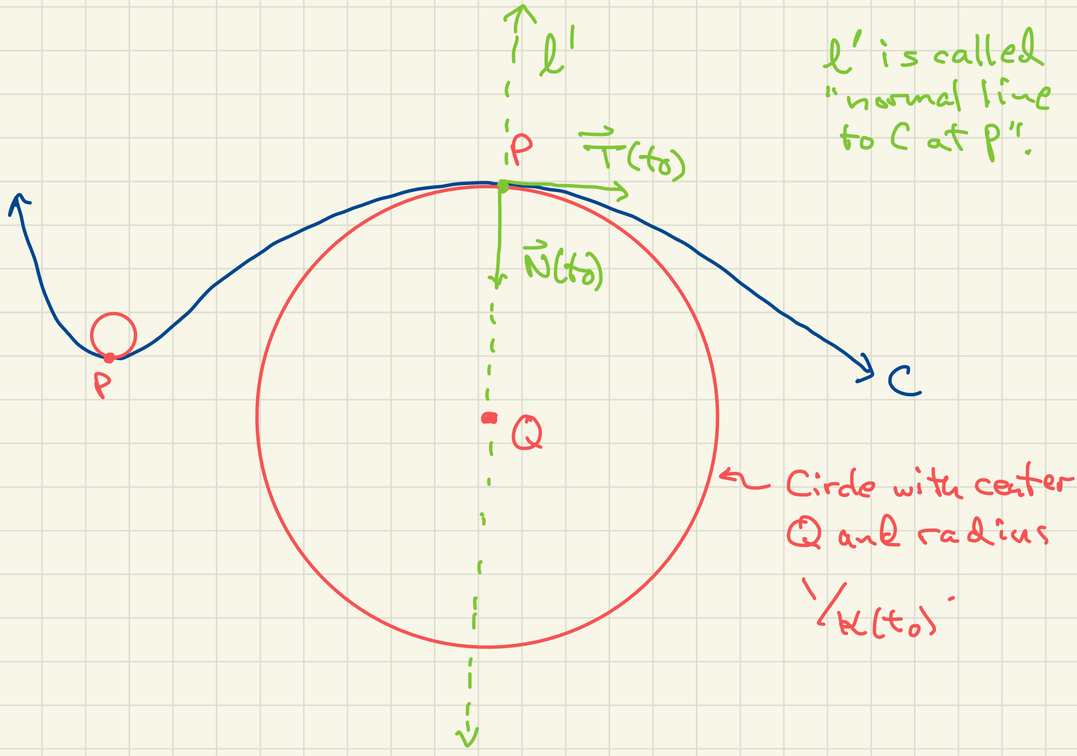
2. Both $\vec{N}(t_0)$ and $\vec{B}(t_0)$ are only defined for values $t = t_0$ where $\vec{T}\,'(t_0) \neq \vec{0}$.

For generic curve $C: \vec{r} = \vec{r}(t)$, $P = \vec{r}(t_0)$:

- $\kappa(t_0)$ measures the "curviness" of C at P.

- The plane $\mathcal{P}$ containing P and having normal vector $\vec{B}(t_0) = \vec{T}(t_0) \times \vec{N}(t_0)$ is the plane which comes closest to containing C near P. (called osculating plane)

- The circle thru P which most closely approximates C near P is contained in the osculating plane at P. It has radius $1/\kappa(t_0)$ and its center is on the line ℓ' through P with direction vector $\vec{N}(t_0)$:



More Brief Interpretations

Let P be the point on C with t-value $t = t_0$.

- If $\vec{r}\,'(t_0) \neq \vec{0}$ then $\vec{r}\,'(t_0)$ is a direction vector for the line tangent to C at P.
  (Often called a "tangent vector at P".)

- If $\vec{r}\,'(t_0) \neq \vec{0}$ then $\vec{T}(t_0)$ is a unit vector which is also a direction vector for tangent at P.

- If $\vec{T}\,'(t_0) \neq \vec{0}$ then
  $\vec{N}(t_0)$ is a unit vector perpendicular to the tangent line at P. It points towards the center of the "osculating circle at P" which is the circle that most closely approximates C near P.

- $\vec{B}(t_0)$ is a normal vector for the osculating plane at P.

- The radius of the osculating circle at P is the reciprocal of $\kappa(t_0)$.

<u>Example</u> Find scalar equations for the line ℓ tangent to the curve

$$C : \vec{r} = \langle \ln(t), 2t, t^2 \rangle \quad , t > 0$$

at the point P where $t = 1$. Also calculate $\vec{T}(t)$.

(Here $t_0 = 1$. we must first calculate $\vec{r}\,'(t)$ then plug in $t = 1$ to get direction vector $\vec{r}\,'(1)$ for ℓ.)

$$\vec{r}\,'(t) = \langle \frac{1}{t}, 2, 2t \rangle$$

So $\vec{r}\,'(1) = \langle 1, 2, 2 \rangle = \vec{d}_\ell$, a direction vector for ℓ.

The point P has coordinates $(0, 2, 1)$ so

$$\ell : \begin{cases} x = t \\ y = 2t + 2 \\ z = 2t + 1 \end{cases}$$

<u>Comment</u>: It may not be necessary to simplify here. But notice how much nicer the answer is.

To determine $\vec{T}(t)$:

$$|\vec{r}\,'(t)| = \left(\frac{1}{t^2} + 4 + 4t^2\right)^{1/2} = \left(\frac{1 + 4t^2 + 4t^4}{t^2}\right)^{1/2}$$

$$= \left(\frac{(1+2t^2)^2}{t^2}\right)^{1/2} \overset{*}{=} \frac{1+2t^2}{t}$$

$$\vec{T}(t) = \frac{t}{1+2t^2}\left\langle \frac{1}{t}, 2, 2t \right\rangle = \frac{1}{1+2t^2}\langle 1, 2t, 2t^2 \rangle$$

* Since $t > 0$ we have $(t^2)^{1/2} = |t| = t$

Example (Stewart #15, page 922)

For the point $P=\langle 0,\pi,1\rangle$ on curve C.

$$C: \begin{cases} x = \sin 2t \\ y = t \\ z = \cos 2t \end{cases}$$

Find (a) tangent line ℓ at P (b) curvature at P (c) osculating plane p at P.

approach: P is the point on C with $t_0 = \pi$.

First calculate all of the key constructs for C at t.

Then take $t = t_0$ and work out answers to (a), (b), (c).

$$\vec{r}(t) = \langle \sin 2t, t, \cos 2t \rangle$$

$$\vec{r}\,'(t) = \langle 2\cos 2t, 1, -2\sin 2t \rangle$$

$$v(t) = |\vec{r}\,'(t)| = (4\cos^2 2t + 1 + 4\sin^2 2t)^{1/2} = \sqrt{5}$$

$$\vec{T}(t) = \frac{1}{v}\vec{r}\,' = \frac{1}{\sqrt{5}}\langle 2\cos 2t, 1, -2\sin 2t\rangle$$

$$\vec{T}\,'(t) = \frac{1}{\sqrt{5}}\langle -4\sin 2t, 0, -4\cos 2t\rangle$$

$$|\vec{T}\,'(t)| = \frac{1}{\sqrt{5}}\sqrt{(4\sin 2t)^2 + (-4\cos 2t)^2} = 4/\sqrt{5}$$

$$\kappa(t) = \frac{|\vec{T}\,'|}{v} = \frac{4/\sqrt{5}}{\sqrt{5}} = 4/5$$

$$\vec{N}(t) = \frac{\vec{T}\,'}{|\vec{T}\,'|} = \langle -\sin t, 0, -\cos 2t\rangle$$

$$\vec{B}(t) = \vec{T}\times\vec{N} = \frac{1}{\sqrt{5}}\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2\cos 2t & 1 & -2\sin 2t \\ -\sin 2t & 0 & -\cos 2t \end{vmatrix} = \frac{1}{\sqrt{5}}\langle -\cos 2t, 2, \sin 2t\rangle$$

continued ↘

Now plug in $t_0 = \pi$. $\quad P = (0, \pi, 1)$

$$\vec{r}\,'(\pi) = \langle 2\cos 2\pi, 1, -2\sin 2\pi \rangle = \langle 2, 1, 0 \rangle$$

$$\kappa(\pi) = 4/5$$

$$\vec{B}(\pi) = \frac{1}{\sqrt{5}} \langle -\cos 2\pi, 2, \sin 2\pi \rangle = \frac{1}{\sqrt{5}} \langle -1, 2, 0 \rangle$$

Answers

(a) $\vec{d}_\ell = \langle 2, 1, 0 \rangle \Rightarrow \ell : \begin{cases} x = 2t \\ y = t + \pi \\ z = 0t + 1 \end{cases}$

or (in vector form) $\vec{r}_\ell(t) = \langle 2t, t+\pi, 1 \rangle$

(b) $\kappa(\pi) = 4/5$

parallel to $\vec{B}(\pi)$

(c) A normal vector for P is $\langle -1, 2, 0 \rangle$. So an equation for P is $-1(x-0) + 2(y-\pi) + 0(z-1) = 0$ which becomes

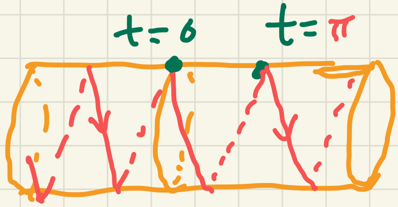
$-x + 2y = 2\pi$. $\leftarrow$ no z in equation indicates this is a vertical plane.

A very rough picture:

note that, for C,

$x^2 + z^2 = \sin^2 2t + \cos^2 2t = 1$

so C lies on the (orange) cylinder $x^2 + z^2 = 1$



About acceleration

The vector function $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ represents the motion of an object in 3-space whose acceleration vector at time t is $\vec{r}\,''(t)$.

How does $\vec{r}\,''(t)$ relate to the key constructs of the curve C determined by $\vec{r}(t)$??

$$\vec{T} = \frac{\vec{r}\,'}{v}$$

$$\vec{r}\,''(t) = \frac{d}{dt}\left[\vec{r}\,'(t)\right] = \frac{d}{dt}\left[v\vec{T}\,\right]$$

$$= v'\vec{T} + v\vec{T}\,' = v'\vec{T} + v|\vec{T}\,'|\frac{\vec{T}\,'}{|\vec{T}\,'|}$$

$$= v'\vec{T} + v|\vec{T}\,'|\vec{N} = v'\vec{T} + v^2\kappa\vec{N}$$

product rule

$$\vec{N} = \frac{\vec{T}\,'}{|\vec{T}\,'|} \qquad \kappa = \frac{|\vec{T}\,'|}{v}$$

Conclusion:

$$\boxed{\vec{r}\,''(t) = v'(t)\vec{T}(t) + v(t)^2\kappa(t)\vec{N}(t)}$$

(recall that $v(t)$ = speed at time t = $|\vec{r}\,'(t)|$ and $\kappa(t)$ = curvature of C at point on C with time value t.)