recap on .....

Geometry of a Curve C — Key Constructs

C: $\vec{r} = \vec{r}(t) = \langle f(t), g(t), h(t) \rangle$

$\vec{r}\,'(t) = \langle f'(t), g'(t), h'(t) \rangle$ = velocity

$|\vec{r}\,'(t)| = \left(f'(t)^2 + g'(t)^2 + h'(t)^2 \right)^{1/2}$ $= v(t)$ = speed

$\vec{T}(t) = \dfrac{1}{|\vec{r}\,'(t)|}\,\vec{r}\,'(t)$ = unit tangent vector

$\vec{T}\,'(t)$

$|\vec{T}\,'(t)|$

$\kappa(t) = \dfrac{|\vec{T}\,'(t)|}{|\vec{r}\,'(t)|}$ = curvature

$\vec{N}(t) = \dfrac{1}{|\vec{T}\,'(t)|}\,\vec{T}\,'(t)$ = unit normal vector

$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$ = unit bi-normal vector

CAUTION: Must watch out for cusp points!

1. Both $\vec{T}(t_0)$ and $\kappa(t_0)$ are only defined for values $t = t_0$ where $\vec{r}\,'(t_0) \neq \vec{0}$.

2. Both $\vec{N}(t_0)$ and $\vec{B}(t_0)$ are only defined for values $t = t_0$ where $\vec{T}\,'(t_0) \neq \vec{0}$.

Comments

(i) To determine an attribute of a curve $C: \vec{r} = \vec{r}(t)$ at point P where $t = t_0$, you first need to work down the list of key constructs for an arbitrary t and then plug in $t = t_0$. Specifically

- To find tangent line at P, determine $\vec{r}\,'(t)$ then plug in $t = t_0$.
- To find curvature at P, calculate $\vec{r}\,'(t)$, $v(t)$, $\vec{T}(t)$, $\vec{T}\,'(t)$, $|\vec{T}\,'(t)|$, $\kappa(t)$, then plug in $t = t_0$
- To find osculating plane, determine $\vec{r}\,'(t)$, $v(t)$, $\vec{T}(t)$, $\vec{T}\,'(t)$, $|\vec{T}\,'(t)|$, $\vec{N}(t)$, $\vec{B}(t)$ — then plug in $t = t_0$

(ii) In working down the list of key constructs for a curve C successive computations often get complicated. At each stage make an effort to simplify the result before moving to next stage.

E.G – See example on page 4 of Notes_12.3.

(iii) Watch out for cusp points.

E.G. – In problem 5 on exam 4, if P is chosen to be the point where $t = 0$ then $\vec{r}\,'(0) = \vec{0}$, and there is a cusp at P. On this exam, this would suggest re-choosing P to be a point on C where $\vec{r}\,'(t_0) \neq \vec{0}$.

↳ Read the discussion of this given in the posted solutions for Exam 4.

More Comments on Exam 4

Questions #1 and #3 involve very important basic problems that need to be well understood.

#1: Fact Two non-equal points uniquely determine a line.

problem: Given $P \neq Q$ find equations for line ℓ containing them.

approach: Use $\overrightarrow{PQ}$ as a direction vector and P as a point on ℓ.

#3: Fact Three non-collinear points uniquely determine a plane p.

problem: Given non-collinear P, Q, R find equation for plane p.

approach: Use $\overrightarrow{PQ} \times \overrightarrow{PR}$ as a normal vector and P as a point on p.

more general idea: If $\vec{u}$ and $\vec{v}$ are vectors parallel to p then $\vec{u} \times \vec{v}$ is a normal vector for p.

There's another approach that can be used for #3:

example Find an equation for plane thru $(1,0,1)$, $(-1,1,0)$ and $(0,1,1)$.

solution Suppose $Ax + By + Cz + D = 0$ is the desired equation. Then

$$\begin{cases} A + C + D = 0 & \text{①} \leftarrow \text{because } (x,y,z) = (1,0,1) \text{ is on } p. \\ -A + B + D = 0 & \text{②} \leftarrow \text{because } (-1,0,1) \text{ is on } p. \\ B + C + D = 0 & \text{③} \leftarrow \text{because } (0,1,1) \text{ is on } p. \end{cases}$$

now solving these equations simultaneously results in:

$$B = A, D = 0, C = -A$$

So p has equation $Ax + Ay - Az = 0 \implies p: x + y - z = 0$

Exam 4 comments, continued

<u>Always</u> look for ways to check your answers!

<u>example</u> We determined that $x+y-z=0$ was an equation for plane containing $(1,0,1)$, $(-1,1,0)$, $(0,1,1)$.

check $(1,0,1)$: $(1)+(0)-(1)=0$ ✓

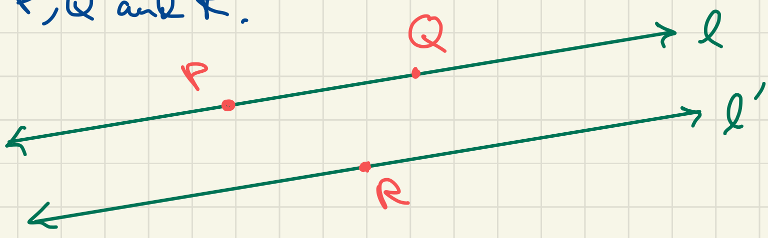
check $(-1,1,0)$: $(-1)+(1)-0=0$ ✓

check $(0,1,1)$: $(0)+(1)-(1)=0$ ✓

If you're unsure how to solve a particular problem try to reduce it to a problem you do know how to solve:

<u>Problem</u> Find an equation for the plane p containing two given parallel lines ℓ and ℓ'.

<u>possible approach</u> Use given descriptions to determine two points $P \neq Q$ on ℓ and a point R on ℓ'. Then p is the (unique) plane containing P, Q and R. Now solve the problem by finding equation for plane thru P, Q and R.



Surfaces in 3-space (primary object of study in Calculus 4)

The graph in 3-space of an equation $F(x,y,z)=0$ of three variables consists of all points (x,y,z) for which the equation $F(x,y,z)=0$ is true.
Typically this will be a surface in 3-space (ie— a 2-dimensional set).

Examples

① The graph of $Ax+By+Cz+D=0$ is a plane in 3-space (provided that at least one of A, B, C is non-zero).

② The graph of $(x-2)^2+(y+1)^2+(z-3)^2=25$ is a sphere. Specifically its the sphere with radius 5 centered at $(2,-1,3)$.

explanation The distance from (x,y,z) to $(2,-1,3)$ is $\sqrt{(x-2)^2+(y+1)^2+(z-3)^2}$, so the described sphere consists of all points where this distance equals 5. Squaring this equation gives $(x-2)^2+(y+1)^2+(z-3)^2=25$.

Observe If we expand this equation we get

$$x^2+y^2+z^2-4x+2y-6z-11=0.$$

It wouldn't be so easy to recognize this as a sphere (unless you complete the squares).

Both examples ① and ② are special cases of a more general family of surfaces described in Chapter 12.

A <u>quadric surface</u> (also called "quadratic surface") is the graph of an equation $F(x,y,z)=0$ where $F(x,y,z)$ is degree 2 polynomial in 3 variables x,y,z.

This means that a quadric surface has an equation:

$$\underbrace{Ax^2+By^2+Cz^2+Dxy+Exz+Fyz}_{\text{degree 2 terms}}+\underbrace{Gx+Hy+Iz}_{\substack{\text{linear terms}\\ \text{(degree 1)}}}+\underbrace{J}_{\substack{\text{constant}\\ \text{term}}}=0$$

The possibilities for these surfaces seems daunting, but it can be shown that there is a short list of possible types that these quadric surfaces can have. To describe this its best to first examine the analogs in 2-space.

In the xy-plane a <u>conic section</u> is the graph of an equation $g(x,y)=0$ where $g(x,y)$ is a degree two polynomial in 2-variables x and y:

$$\underbrace{Ax^2+Bxy+Cy^2}_{\text{degree 2}}+\underbrace{Dx+Ey}_{\text{linear}}+\underbrace{F}_{\text{constant}}=0$$

Notes on Conic Sections

A curve in the xy-plane that is the graph of a quadratic equation of two variables

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \tag{1}$$

is called a **conic** or **conic section**.

These curves are classified by their type (either ellipse, parabola, or hyperbola) and their degeneracy (either non-degenerate or degenerate). The type can be read off from the equation (1) by looking at the value of $B^2 - 4AC$. If $B^2 - 4AC < 0$ then the conic is a (possibly degenerate) ellipse, if $B^2 - 4AC = 0$ then it is a (possibly degenerate) parabola, and if $B^2 - 4AC > 0$ then it is a (possibly degenerate) hyperbola.

Each of the non-degenerate conics has a "standard form" equation as follows:

- **ellipse:** $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ where $a > 0$ and $b > 0$

  This ellipse has center at the origin $(0, 0)$, is symmetric across both the x- and the y-axes, and has axis lengths of $2a$ and $2b$. It is a circle (with diameter $2a$) whenever $a = b$.

- **parabola:** $y = cx^2$ where $c \neq 0$.

  This parabola has vertex at the origin $(0, 0)$, is symmetric across the y-axis, and has its "focus" at the point $(0, c/4)$. ← Correction: The focus for this parabola $y = cx^2$ is $(0, \frac{1}{4c})$.

- **hyperbola:** $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ where $a > 0$ and $b > 0$

  This hyperbola has center at the origin $(0, 0)$, is symmetric across both the x- and the y-axes, and has asymptotes $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$ (same as, $y = \pm bx/a$).

The term "standard form" indicates that

> *Each non-degenerate conic with equation (1) can be moved via translation and/or rotation of the xy-plane to a conic with a standard form equation.*

In other words each conic is isometric (that is, has the same shape) as a standard form conic section. Each of the non-degenerate conics has additional, more detailed geometric attributes, some of which you can read about in sections 10.5 and 10.6, and Appendix C of Stewart.

The degenerate conics are usually easy to recognize. They can be identified by their graphs which have one of the following forms:

- **degenerate ellipse:** the empty set, or, a point

  Examples: $x^2 + y^2 + 1 = 0$ is the empty set, $x^2 + y^2 = 0$ is a single point

- **degenerate parabola:** the empty set, a straight line, or, a pair of parallel lines

  Examples: $x^2 + 1 = 0$ is the empty set, $x^2 = 0$ is a line (the y-axis), $x^2 - 1 = 0$ is a pair of parallel lines (the vertical lines $x = 1$ and $x = -1$).

- **degenerate hyperbola:** a pair of intersecting lines

  Examples: $x^2 - y^2 = 0$ is a pair of intersecting lines (the lines $y = x$ and $y = -x$).

Problem: For $a>0$ and $b>0$,

The graph of the polar coordinate equation

$$r=\frac{c}{a+b\cos\theta}$$

is a conic section. Which type is it?

To answer, let's convert to a rectangular equation:

$$r=\frac{c}{a+b\cos\theta} \Rightarrow ar+br\cos\theta=c$$

$$\Rightarrow a\sqrt{x^2+y^2}+bx=c \Rightarrow a\sqrt{x^2+y^2}=c-bx$$

$$\Rightarrow a^2(x^2+y^2)=(c-bx)^2=c^2-2bx+b^2x^2$$

$$\Rightarrow (a^2-b^2)x^2+a^2y^2+2bx-c^2=0$$

Then $B^2-4AC=0^2-4(a^2-b^2)a^2$ which is

$$\begin{cases}\text{negative} & \text{when } a>b\\ 0 & \text{when } a=b\\ \text{positive} & \text{when } a<b\end{cases}$$

ANSWER

$$\begin{cases}\text{Ellipse when } a>b\\ \text{Parabola when } a=b\\ \text{Hyperbola when } a<b\end{cases}$$