Notes on Conic Sections

A curve in the $xy$-plane that is the graph of a quadratic equation of two variables

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \tag{1}$$

is called a **conic** or **conic section**.

These curves are classified by their type (either ellipse, parabola, or hyperbola) and their degeneracy (either non-degenerate or degenerate). The type can be read off from the equation (1) by looking at the value of $B^2 - 4AC$. If $B^2 - 4AC < 0$ then the conic is a (possibly degenerate) ellipse, if $B^2 - 4AC = 0$ then it is a (possibly degenerate) parabola, and if $B^2 - 4AC > 0$ then it is a (possibly degenerate) hyperbola.

Each of the non-degenerate conics has a "standard form" equation as follows:

- **ellipse:** $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ where $a > 0$ and $b > 0$

  This ellipse has center at the origin $(0, 0)$, is symmetric across both the $x$- and the $y$-axes, and has axis lengths of $2a$ and $2b$. It is a circle (with diameter $2a$) whenever $a = b$.

- **parabola:** $y = cx^2$ where $c \neq 0$.

  This parabola has vertex at the origin $(0, 0)$, is symmetric across the $y$-axis, and has its "focus" at the point $(0, c/4)$. ← Correction The focus for the parabola $y = cx^2$ is $(0, \frac{1}{4c})$.

- **hyperbola:** $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ where $a > 0$ and $b > 0$

  This hyperbola has center at the origin $(0, 0)$, is symmetric across both the $x$- and the $y$-axes, and has asymptotes $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$ (same as, $y = \pm bx/a$).

The term "standard form" indicates that

> *Each non-degenerate conic with equation (1) can be moved via translation and/or rotation of the xy-plane to a conic with a standard form equation.*

In other words each conic is isometric (that is, has the same shape) as a standard form conic section. Each of the non-degenerate conics has additional, more detailed geometric attributes, some of which you can read about in sections 10.5 and 10.6, and Appendix C of Stewart.

The degenerate conics are usually easy to recognize. They can be identified by their graphs which have one of the following forms:

- **degenerate ellipse:** the empty set, or, a point

  Examples: $x^2 + y^2 + 1 = 0$ is the empty set, $x^2 + y^2 = 0$ is a single point

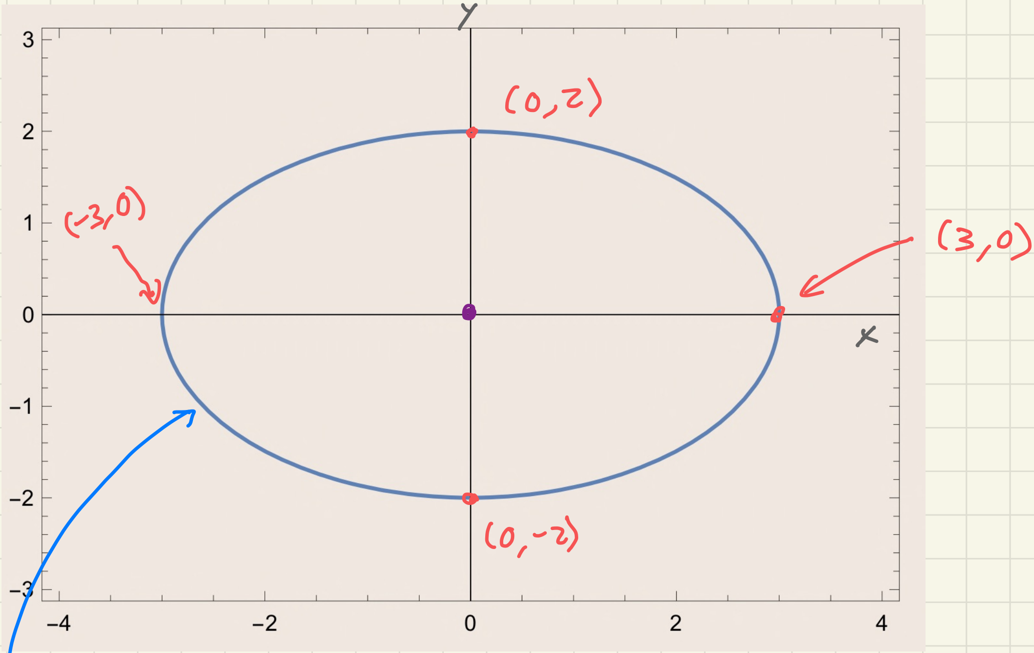
- **degenerate parabola:** the empty set, a straight line, or, a pair of parallel lines

  Examples: $x^2 + 1 = 0$ is the empty set, $x^2 = 0$ is a line (the $y$-axis), $x^2 - 1 = 0$ is a pair of parallel lines (the vertical lines $x = 1$ and $x = -1$).

- **degenerate hyperbola:** a pair of intersecting lines

  Examples: $x^2 - y^2 = 0$ is a pair of intersecting lines (the lines $y = x$ and $y = -x$).

# Standard Form Ellipse




$4x^2 + 9y^2 = 36$

Dividing by 36 gives $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$ so this is a standard form ellipse with $a = 3$, $b = 2$.

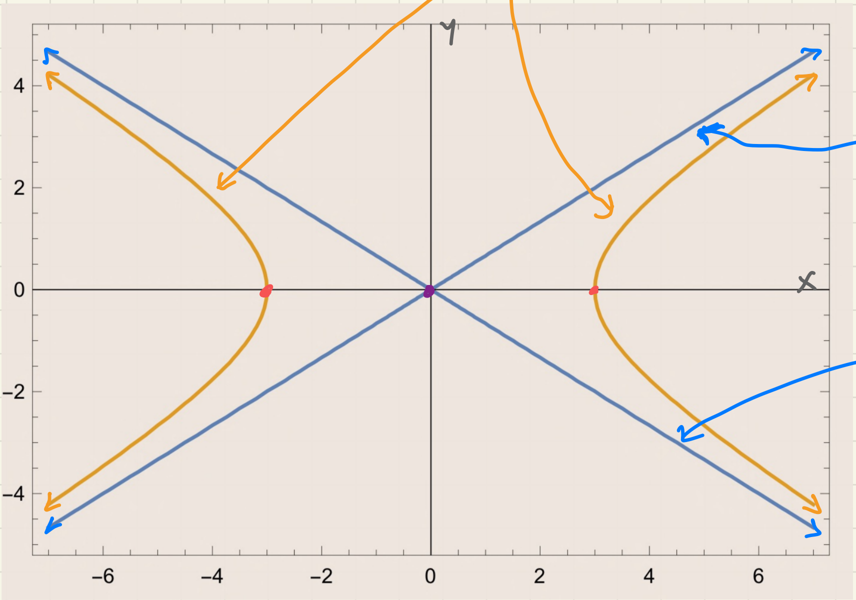
vertices: $(-3,0)$ and $(3,0)$

center: $(0,0)$

axes of symmetry: $y = 0$, $x = 0$

# Standard Form Hyperbola

$4x^2 - 9y^2 = 36$




$y = \frac{2}{3}x$

$y = -\frac{2}{3}x$

center: $(0,0)$

vertices: $(-3,0)$ and $(3,0)$

axes: $x=0$ and $y=0$

asymptotes: $4x^2 - 9y^2 = 0 \Rightarrow y = \pm\frac{2}{3}x$

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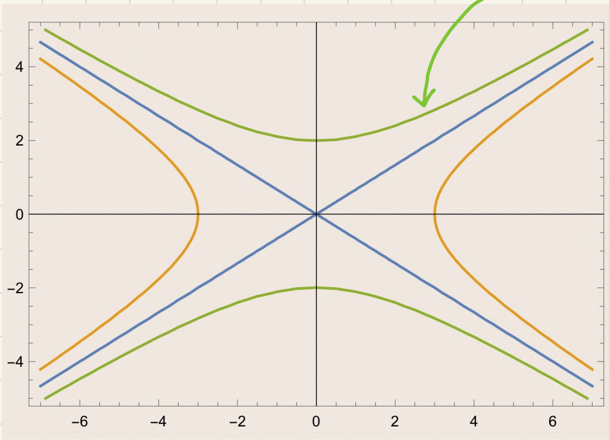
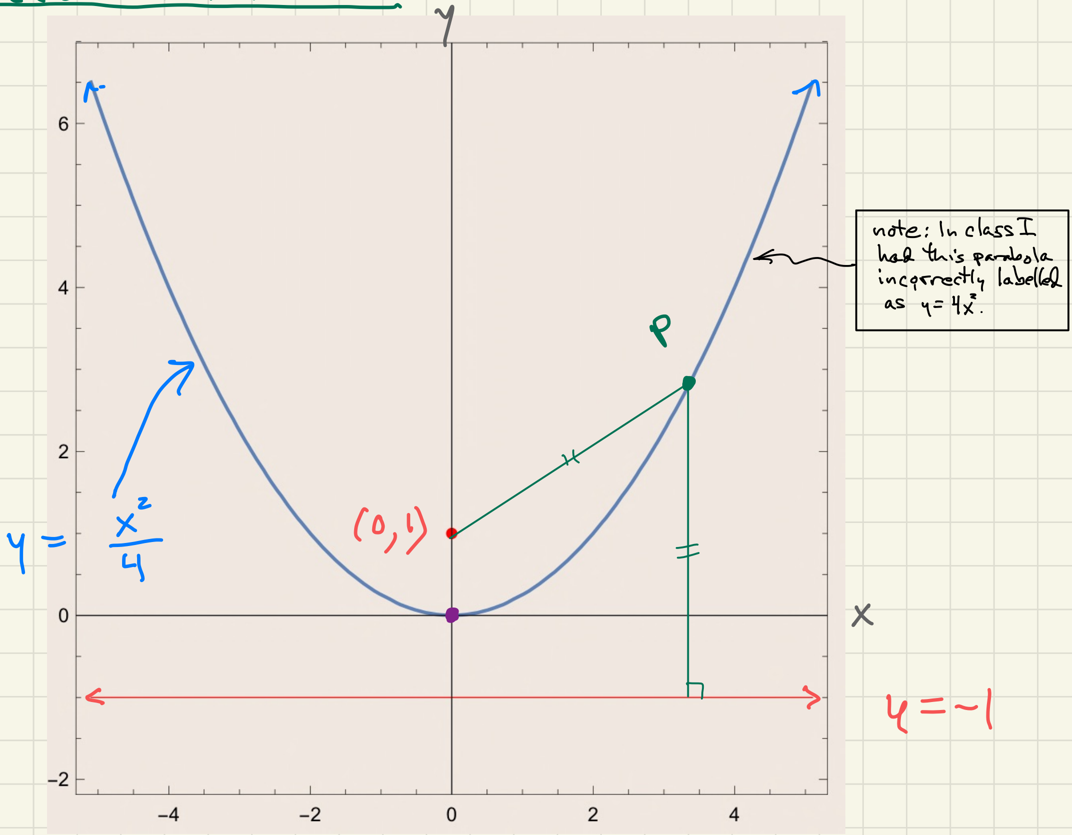
$4x^2 - 9y^2 = -36$




(Hyperbola with same asymptotes $y = \pm\frac{2}{3}x$.)

# Standard Form Parabola




vertex: $(0,0)$

axis of symmetry: $x=0$ (y-axis)

focus: $(0,1)$

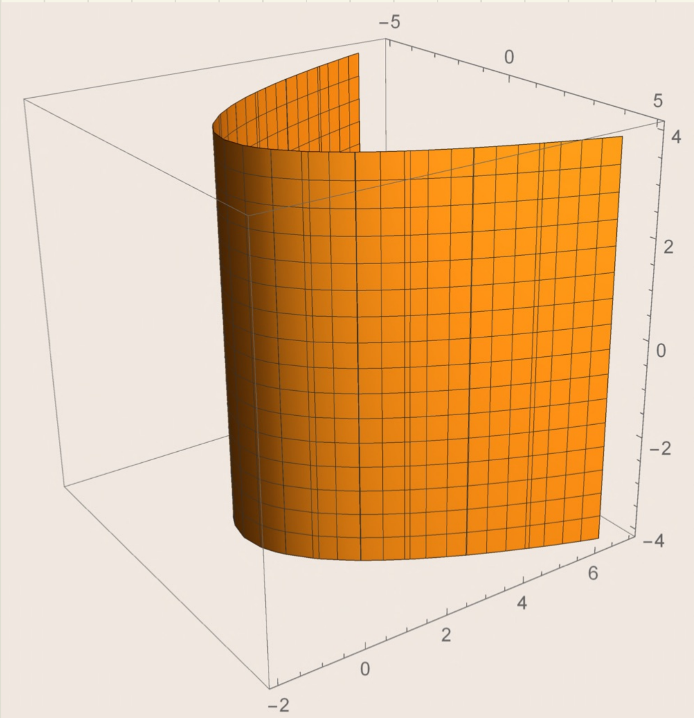
directrix: $y=-1$ (horizontal line)

geometric fact: The distance from any point P on the parabola to the focus equals its distance to the directrix.

example What does the graph $S$ of $y=\frac{1}{4}x^2$ in $xyz$-space look like?

Answer Its graph in the $xy$-plane is a parabola but its graph in $xyz$-space is a "parabolic cylinder".

Discussion If $(x,y,0)$ is a point on the surface $S$ then $(x,y,z)$ is also on $S$ for any value of $z$. This means that the entire vertical line through $(x,y,0)$ is on the surface $S$. Since the set of points $(x,y,0)$ where $y=\frac{1}{4}x^2$ is a parabola in the $xy$-plane then $S$ is the vertical "cylinder" consisting of all of the vertical lines through points on the parabola.




(More discussion of cylinders is included below.)

QUADRIC SURFACES:

A quadric surface (or quadratic surface) is the graph in 3-space of a quadratic equation in three variables

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0. \tag{1}$$

As with the conic sections, each quadric surface can be classified into one of a few different primary types, and may possibly have "degenerate" form. The non-degenerate conic sections form an important class of examples that will be very useful in the general study of surfaces which forms the focal point in the fourth semester calculus class.

The graph of a quadric surface with degenerate form is either the empty set, a single point, a plane, or a cylinder over a conic section. For example, the graph in the 3-space of the equation $x^2 + y^2 + z^2 = k$ has degenerate form whenever $k \leq 0$: if $k < 0$ then its graph is the empty set, and if $k = 0$ its graph is a single point. For $k > 0$, this surface $x^2 + y^2 + z^2 = k$ is non-degenerate (it is a sphere, which is a special type of ellipsoid).

The non-degenerate quadric surfaces are more interesting and important. It can be shown that each of these surfaces can be moved via translation and rotation in 3-space to have one of six 'standard forms', as shown on the next page. The name of each of these surfaces can be easily remembered by looking at the three "coordinate traces" of the surface.

The **coordinate traces of a surface** $S$ are the intersections of the surface with the three coordinate planes: $z = 0$ ($xy$-plane), $y = 0$ ($xz$-plane) or $x = 0$ ($yz$-plane). Each of the coordinate traces of a quadric surface will be a (possibly degenerate) conic section. For example, the coordinate traces for an elliptic paraboloid consist of two parabolas (hence it is called a 'paraboloid') and one ellipse (hence it is called elliptic).

More generally, the **trace of a surface** $S$ **in a plane** $\mathcal{P}$ is the curve of intersection of the surface $S$ and the plane $\mathcal{P}$. For a quadric surface $S$, each of these traces will be a conic section in the plane $\mathcal{P}$. It is not difficult to show that parallel planes always give rise to conic sections of the same type although one might be non-degenerate while the other is degenerate.

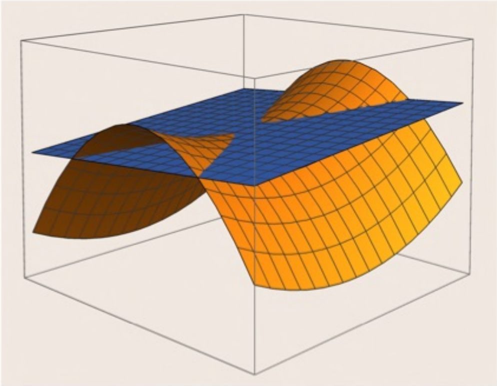
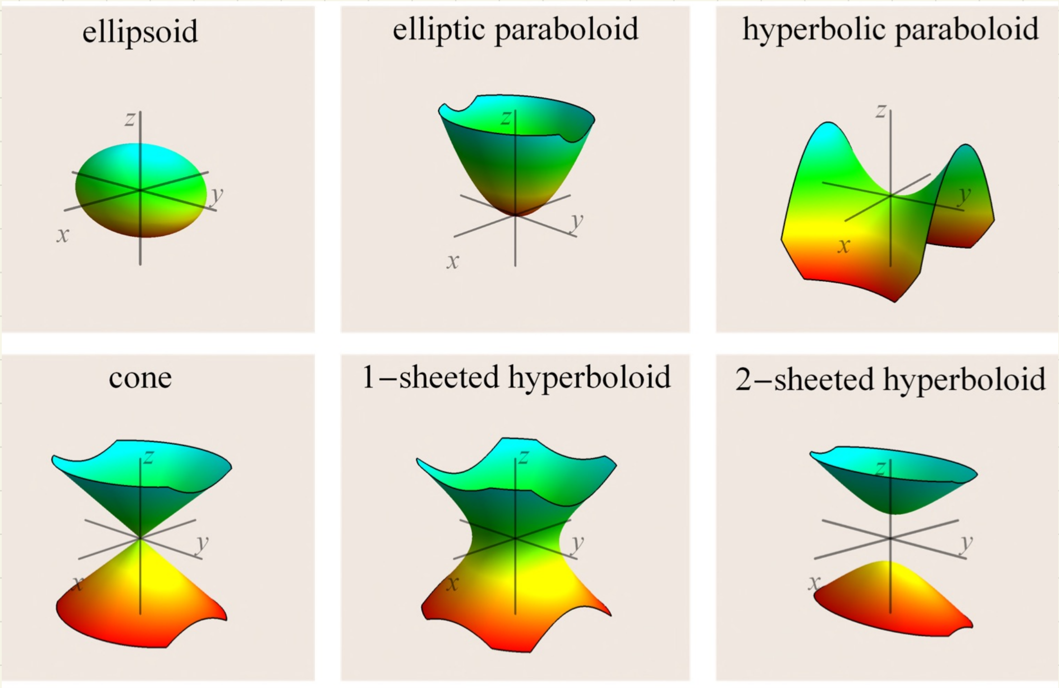


Figure 1: *The trace of the pictured hyperboloid is a non-degenerate hyperbola, but if you move the plane down slightly the trace will become a degenerate hyperbola (pair of intersecting lines).*

All six of the quadric surfaces exhibit some symmetry. The standard form ellipsoids, cones and hyperboloids of 1- or 2-sheets are symmetric across all three coordinate planes. (This is because replacing either $x$, $y$ or $z$ by its negative does not change the equation. So if $(x, y, z)$ is on the surface then all eight points $(\pm x, \pm y, \pm z)$ are also on the surface.) The two standard form paraboloids are symmetric across the $xz$- and $yz$-coordinate planes. (But they are not symmetric across the $xy$-plane, as can easily be seen in figure 2.) Properties of the quadric surfaces are described in more detail in section 12.6 of Stewart's book, and even more on the Wikipedia page entitled "quadrics".

There are 6 types of non-degenerate quadric surfaces:




These surfaces are in standard form and have equations as indicated, where $a>0$, $b>0$ and $c>0$:

- ellipsoid: $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$; traces: 3 ellipses
- elliptic paraboloid: $\dfrac{z}{c}=\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}$; traces: 2 parabolas, 1 degenerate ellipse;
- hyperbolic paraboloid: $\dfrac{z}{c}=\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}$; traces: 2 parabolas, 1 degenerate hyperbola
- cone: $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}=0$; traces: 2 degenerate hyperbolas, 1 degenerate ellipse
- hyperboloid of one sheet: $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}=1$; traces: 2 hyperbolas, 1 ellipse
- hyperboloid of two sheets: $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}=-1$; traces: 2 hyperbolas, 1 degenerate ellipse

example $6x^2 + 4y^2 - 3z^2 = 12$ has coordinate traces:

xy-trace : $z = 0 \Rightarrow 6x^2 + 4y^2 = 12$ ellipse in xy-plane

xz-trace : $y = 0 \Rightarrow 6x^2 - 3z^2 = 12$ hyperbola in xz-plane

yz-trace : $x = 0 \Rightarrow 4y^2 - 3z^2 = 12$ hyperbola in yz-plane

This surface is an elliptic hyperbola.

(same as $\frac{x^2}{2} + \frac{y^2}{3} - \frac{z^2}{4} = 1$ )

**example 1.** The graph of the equation $16x^2 + 25y^2 - 4z = 0$ is a standard form elliptic paraboloid since its equation can be written as $\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ where $a = 1/4$, $b = 1/5$ and $c = 1/4$. The quadric surface $16z^2 + 25y^2 - 4x = 0$ is also an elliptic paraboloid since it comes from the previous equation by simply interchanging the variables $x$ and $y$. However it opens up along the $x$-axis rather than the $z$-axis. Technically this is no longer a standard form quadric surface because the equation doesn't have the required form. But you can see that any of the standard form quadric surfaces will determine up to six different quadric surfaces by permuting the variables $x$, $y$ and $z$.

**example 2.** *Is the graph of $x^2 + 3y^2 - 4z^2 + 6y - 4z = 1$ a quadric surface? If so, which of the six types does it have?*
We use the process of completing the square to rewrite the left hand side of the equation as

$$x^2 + 3y^2 - 4z^2 + 6y - 4z = x^2 + 3(y^2 + 2y + 1) - 3 - 4(z^2 - z + \frac{1}{4}) + 1 = x^2 + 3(y+1)^2 - 4(z - \frac{1}{2})^2 - 2\ .$$

So the equation becomes $x^2 + 3(y+1)^2 - 4(z - 1/2)^2 = 3$ which can be expressed as

$$\frac{x^2}{3} + \frac{(y+1)^2}{1} - \frac{(z-1/2)^2}{3/4} = 1.$$

This surface is a translation of

$$\frac{x^2}{3} + \frac{y^2}{1} - \frac{z^2}{3/4} = 1$$

which is the standard form equation for a 1-sheeted hyperboloid (with $a = \sqrt{3}$, $b = 1$ and $c = \sqrt{3}/2$).

CYLINDERS:

Let $C$ be a curve in the $xy$-plane that is the graph of an equation $E(x, y) = 0$ of two variables. If this equation is considered to be an equation in three variables $x$, $y$ and $z$, then it also determines a graph in $xyz$-space, which is called a "cylindrical surface". The intersection of this surface with the $xy$-coordinate plane consists of all points $(x, y, 0)$ where $(x, y)$ lies on the curve $C$, and, in this way, we can think of $C$ as being the curve of intersection of plane $z = 0$ with the cylindrical surface. Notice that if the point $(x, y, 0)$ is on $C$ then, for every choice of real number $z$, $(x, y, z)$ will be on the cylindrical surface (because the variable $z$ doesn't appear in the equation $E(x, y) = 0$ for the surface). This shows that the cylindrical surface consists entirely of the vertical lines above (and below) the points in $C$. We refer to a cylindrical surface constructed in this way as a **vertical cylinder** or as a **cylinder in the direction of the $z$-axis**. Thus a vertical cylinder is the graph of an equation of three variables $x, y, z$ in which $z$ does not appear.

If a surface in $xyz$-space has an equation in which either $y$ or $x$ does not appear then that surface will be a cylinder in the direction of the $y$-axis or $x$-axis respectively. More generally, if $\ell$ is any line in 3-space and $C$ is a curve in a plane perpendicular to $\ell$ then the collection of all lines which intersect $C$ and are parallel to $\ell$ comprise a **cylinder in the direction of $\ell$**. Any cylinder in 3-space can be obtained by rotating a vertical cylinder.

Some simple examples of cylinders are:

- $(x - a)^2 + (y - b)^2 = R^2$ is a "vertical circular cylinder of radius $R$" (this is the standard usage of the term "cylinder").
- If $C$ is a straight line in the $xy$-plane then the vertical cylinder over $C$ is a plane in 3-space. We might refer to this as a "vertical" plane because it is parallel to the $z$-axis (and a normal vector for this plane is perpendicular to $\vec{k}$).
- If a plane in 3-space is parallel to a line $\ell$ then that plane will be a cylinder in the direction of $\ell$.
- A cylinder over a curve $C$ which is a conic section will be called an elliptic cylinder, a parabolic cylinder or a hyperbolic cylinder depending on the type of the conic.

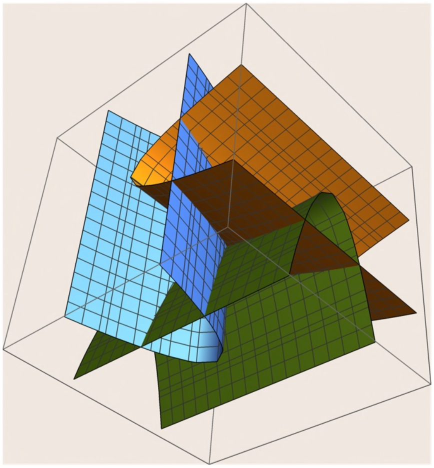


Figure 3: *The intersection of three parabolic cylinders: $y = x^2$, $z = y^2$ and $x = z^2$.*