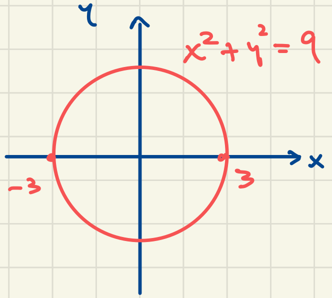
More examples illustrating:

(5) The graph in xyz-space of an equation $F(x,y,z)=0$ with variables x, y, z consists of all points (x,y,z) which satisfy the equation.

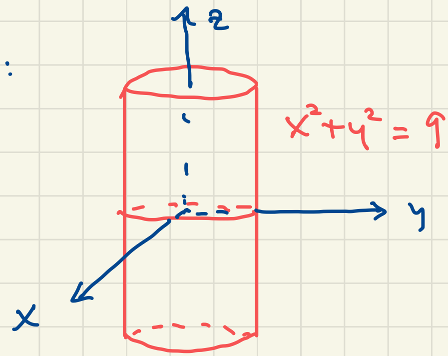
Example 1 What is the graph of $x^2+y^2=9$?

To answer the question we need context ~ are we graphing the equation in the xy-plane, or in xyz-space?

in xy-plane:



in xyz-space:



Example 2 Find the equation for the sphere S of radius 3 centered at $C=(2,-1,-3)$.

S consists of all points $P(x,y,z)$ whose distance from C is 3.

$$3 = \text{dist}\big((x,y,z),(2,-1,-3)\big) = \sqrt{(x-2)^2+(y+1)^2+(z+3)^2}$$

squaring gives a nicer equation

$$\boxed{(x-2)^2+(y+1)^2+(z+3)^2 = 9}$$

more generally:

An equation for the sphere centered at the point $C = (a, b, c)$ and having radius R is:

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = R^2$$

(Note this equation could be multiplied out to get:

$$x^2 + y^2 + z^2 - 2ax - 2bx - 2cx = R^2 - a^2 - b^2 - c^2$$

but this equation is messier and usually would not considered to be an improvement.)

Examples ① What does the graph of

$$(x-2)^2 + (y+1)^2 + (z-3)^2 = -5$$

look like in xyz-space?

Notice that the LHS is the sum of 3 numbers that are 0 or larger, so $LHS \geq 0$ but $RHS = -5 < 0$.

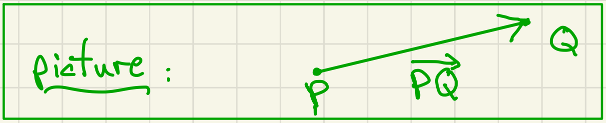
Conclusion, there are no points (x, y, z) satisfying this equation and its graph is the empty set.

② How about $(x-2)^2 + (y+1)^2 + (z-3)^2 = 0$?

well this equation has exactly one solution, namely $(x, y, z) = (2, -1, 3)$, so its graph consists of a single point. (sphere of radius 0 ?)

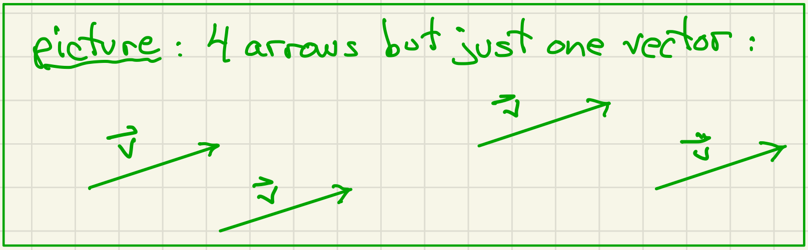
Vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$

If P and Q are points (in either 2-space or 3-space) then the line segment starting at P and ending at Q will be denoted by $\overrightarrow{PQ}$ and called an <u>arrow</u> (or a directed line segment).



Each arrow $\overrightarrow{PQ}$ determines a vector $\vec{v}$ subject to the following condition:

- Two arrows represent the same vector whenever they are parallel, have the same length and point in the same direction.



<u>Note</u>: This is a bit of a technicality but I tend to write $\rightarrow$ for an arrow, and $\rightharpoonup$ for a vector.

<u>Algebraic Viewpoint</u> If $\vec{v}$ is a 3-dimensional vector then there will be an arrow representing $\vec{v}$ starting at the origin $O = (0,0,0)$ and ending at some point $P = (a,b,c)$. In this situation we write $\vec{v} = \langle a,b,c \rangle$ (or $\vec{v} = a\vec{\imath} + b\vec{\jmath} + c\vec{k}$).

Here $\vec{\imath} = \langle 1,0,0 \rangle$, $\vec{\jmath} = \langle 0,1,0 \rangle$, $\vec{k} = \langle 0,0,1 \rangle$

<u>General Principle</u>: If $\vec{v} = \langle a, b, c \rangle$ and $P = (x, y, z)$ then there is one and only one arrow $\overrightarrow{PQ}$ that represents $\vec{v}$ and it ends at the point $Q = (x+a, y+b, z+c)$.

<u>example</u> You can use this principle to determine the algebraic form of the vector $\overrightarrow{PQ}$ if you know the coordinates of P and Q.

For instance, if $P = (-1, -3, 5)$ and $Q = (2, -1, 3)$ then

$\overrightarrow{PQ} = \langle 2-(-1), -1-(-3), 3-5 \rangle = \langle 3, 2, -2 \rangle$ and $\overrightarrow{QP} = \langle -3, -2, 2 \rangle$

There are 5 operations involving vectors that carry a lot of geometric information:

- vector addition
- scalar multiplication
- length (also known as magnitude or norm)
- dot product
- cross product (this only works in dimension 3)

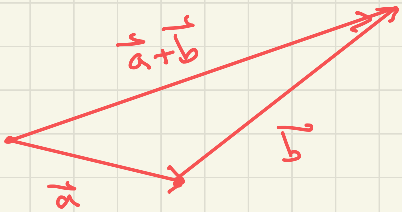
We will examine these one at a time (sections 12.2 thru 12.4 in Stewart).

Vector addition

note input: two vectors
output: one vector

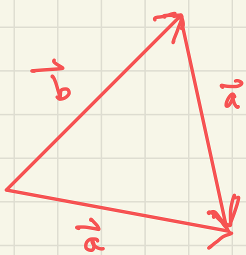
If $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$ then
$\vec{a} + \vec{b} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$.

This definition is geometrically encoded in a triangle:

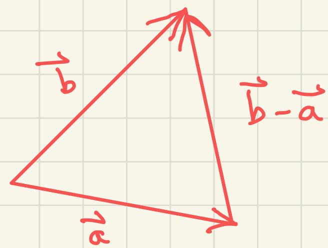


← choose an arrow representing $\vec{b}$ to start at the end point for the arrow representing $\vec{a}$

We can also interpret this as

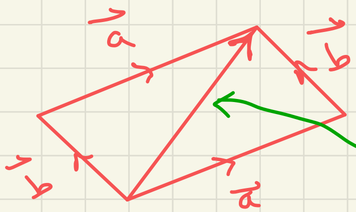


or



where $\langle b_1, b_2, b_3 \rangle - \langle a_1, a_2, a_3 \rangle = \langle b_1 - a_1, b_2 - a_2, b_3 - a_3 \rangle$

The "Commutative Law for Vector Addition" says that
$\vec{a} + \vec{b} = \vec{b} + \vec{a}$ for all vectors, also known as parallelogram law because



$\vec{a} + \vec{b} = \vec{b} + \vec{a}$