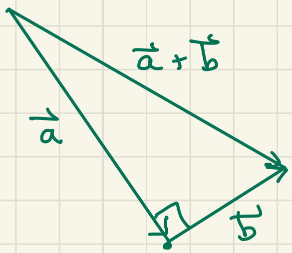
Dot Product

input : 2 vectors
output : a scalar

Question: How can we tell if two vectors $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$ are perpendicular to each other?

If they are perpendicular then we can represent $\vec{a}$ and $\vec{b}$ with arrows as indicated in the picture



← Pythagorus says:
$$|\vec{a}|^2 + |\vec{b}|^2 = |\vec{a}+\vec{b}|^2$$

Observe that $|\vec{a}|^2 = \left(\sqrt{a_1^2 + a_2^2 + a_3^2}\right)^2 = a_1^2 + a_2^2 + a_3^2$ and $|\vec{b}|^2 = b_1^2 + b_2^2 + b_3^2$ and $|\vec{a}+\vec{b}|^2 = (a_1+b_1)^2 + (a_2+b_2)^2 + (a_3+b_3)^2$ so the Pythagorus equation becomes

$$(a_1^2 + a_2^2 + a_3^2) + (b_1^2 + b_2^2 + b_3)^2 = (a_1+b_1)^2 + (a_2+b_2)^2 + (a_3+b_3)^2$$

$$= (a_1^2 + 2a_1b_1 + b_1^2) + (a_2^2 + 2a_2b_2 + b_2^2) + (a_3^2 + 2a_3b_3 + b_3^2)$$

which after cancelling like terms and dividing by 2 gives
$$a_1b_1 + a_2b_2 + a_3b_3 = 0$$

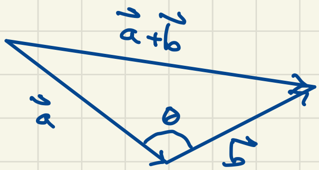
Conclusion $\vec{a}$ and $\vec{b}$ will be perpendicular when
$$a_1b_1 + a_2b_2 + a_3b_3 = 0.$$

Definition The **dot product** of vectors $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$ is the scalar

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 ,$$

$$\boxed{\text{Vectors } \vec{a} \text{ and } \vec{b} \text{ are perpendicular if and only if } \vec{a} \cdot \vec{b} = 0.}$$

The Law of Cosines is a broader version of the Pythagorean Theorem:



$$|\vec{a}|^2 + |\vec{b}|^2 - 2\cos\theta \, |\vec{a}|\,|\vec{b}| = |\vec{a} + \vec{b}|^2$$

Expanding this out in similar fashion as on the previous page produces a very important result:

6 Corollary If θ is the angle between the nonzero vectors **a** and **b**, then

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|}$$

(Stewart page 849)

This can also be written as: $\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos\theta$

Example Notice that if $\theta = \pi/2$ (right angle) then $\cos\theta = 0$ and $0 = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}|\,|\vec{b}|}$ which confirms that $\vec{a}$ and $\vec{b}$ are perpendicular when $\vec{a} \cdot \vec{b} = 0$.

Terminology In mathematics there are numerous synonyms for "perpendicular" two of which are "orthogonal" or "normal".

If $\vec{a}$ and $\vec{b}$ are vectors then

- "$\vec{a} \perp \vec{b}$" means "$\vec{a}$ is perpendicular to $\vec{b}$"
- "$\vec{a} \parallel \vec{b}$" means "$\vec{a}$ is parallel to $\vec{b}$"

page 847:

2 Properties of the Dot Product If **a**, **b**, and **c** are vectors in V_3 and c is a scalar, then

1. $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$
2. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
3. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
4. $(c\mathbf{a}) \cdot \mathbf{b} = c(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (c\mathbf{b})$
5. $\mathbf{0} \cdot \mathbf{a} = 0$

Observe the following: If θ is the angle between vectors $\vec{a}$ and $\vec{b}$, we may assume that θ is between 0 and π and

$$\boxed{\vec{a} \cdot \vec{b} = \begin{cases} \text{positive} & \text{when } 0 \le \theta < \pi/2 \quad \text{(acute)} \\ 0 & \text{when } \theta = \pi/2 \quad \text{(right)} \\ \text{negative} & \text{when } \pi/2 < \theta \le \pi \quad \text{(obtuse)} \end{cases}}$$