Collinear/Coplanar

Familiar Fact 1 In either 2-space or 3-space, any two non-equal points determine a unique line containing them.

Three distinct points are collinear if they lie on a line. (This is possible but statistically unlikely if the points are randomly chosen.)

Problem Given points P, Q, R determine if they are collinear.

Note: The points determine 6 vectors $\vec{PQ}, \vec{QP}, \vec{PR}, \vec{RP}, \vec{QR}, \vec{RQ}$ and three distances $dist(P,Q)$, $dist(P,R)$, $dist(Q,R)$.

method 1 Determine if $\vec{PQ}$ and $\vec{PR}$ are parallel.

method 2 Calculate $\vec{PQ} \times \vec{PR}$. If its equals $\vec{0}$ they are collinear.

method 3 If the sum of two of the distances equals the third then the points are collinear.

Familiar Fact 2 In 3-space any three non-collinear points determine a unique plane.

If four points in 3-space are contained in a plane they are called coplanar. (Again, this is statistically unlikely.)

Problem: Given four points in 3-space determine if whether or not they are coplanar.

method Let P, Q, R, S be 4 points in 3-space, and let $\vec{a} = \overrightarrow{PQ}$, $\vec{b} = \overrightarrow{PR}$, $\vec{c} = \overrightarrow{PS}$.

The cross product $\vec{b} \times \vec{c}$ is perpendicular to both $\vec{b}$ and $\vec{c}$. If the points are coplanar then $\vec{b} \times \vec{c}$ must also be perpendicular to $\vec{a}$. This means that $\vec{a} \cdot (\vec{b} \times \vec{c})$ should equal 0.

Example $P = (0,1,-1)$, $Q = (2,1,1)$, $R = (-3,0,-1)$, $S = (4,4,4)$.

$\vec{a} = \overrightarrow{PQ} = \langle 2,0,2 \rangle$

$\vec{b} = \overrightarrow{PR} = \langle -3,-1,0 \rangle$

$\vec{c} = \overrightarrow{PS} = \langle 4,3,5 \rangle$

check

$\langle -5,15,-5 \rangle \cdot \langle -3,-1,0 \rangle = 15-15 = 0$

$\langle -5,15,-5 \rangle \cdot \langle 4,3,5 \rangle = -20+45-25 = 0$

$$\vec{b} \times \vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -3 & -1 & 0 \\ 4 & 3 & 5 \end{vmatrix} = -5\vec{i} + 15\vec{j} - 5\vec{k}$$

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \langle 2,0,2 \rangle \cdot \langle -5,15,-5 \rangle = -20$$

Conclude: These four points in $\mathbb{R}^3$ are <u>not</u> coplanar.

Parametric Equations

The equations

$$\begin{cases} x = f(t) \\ y = g(t) \end{cases} \qquad a \leq t \leq b$$

describe the motion of an object from time $t=a$ to time $t=b$ in the xy-plane. The function $f(t)$ gives the horizontal position at time t, and $g(t)$ tracks the vertical position.

In these equations, the variable t is called the parameter. The motion of the object traces out a curve C in the xy-plane. We say that the equations $x=f(t)$, $y=g(t)$ give a parametrization of the curve C.

In this situation we say that the point $(f(a), g(a))$ is the initial point of the curve and that $(f(b), g(b))$ is the terminal point. (However we will not always put restrictions on t.)

Basic Problem: Determine geometric information about C from the equations $x=f(t)$, $y=g(t)$.

Example Let's start to find some information about the curve C parametrized by the equations:

$$\begin{cases} x = \cos(t) \\ y = \left(t - \frac{\pi}{2}\right)^2 \end{cases} \qquad -2\pi \leq t \leq 3\pi.$$

① initial point $= \left(1, \frac{25}{4}\pi^2\right)$ when $t = -2\pi$

terminal point $= \left(-1, \frac{25}{4}\pi^2\right)$ when $t = 3\pi$

② Since $-1 \leq \cos t \leq 1$, the curve C is between the two vertical lines $x = -1$ and $x = 1$.

③ The curve intersects $x = -1$ whenever $\cos(t) = -1$. This happens for $t = -\pi, \pi, 3\pi$ and the points are $\left(-1, \frac{9}{4}\pi^2\right), \left(-1, \frac{1}{4}\pi^2\right), \left(-1, \frac{25}{4}\pi^2\right)$

④ C intersects $x = 1$ when $t = -2\pi, 0, 2\pi$. The points are $\left(1, \frac{25}{4}\pi^2\right), \left(1, \frac{1}{4}\pi^2\right), \left(1, \frac{9}{4}\pi^2\right)$.

⑤ Since $(t - \pi/2)^2 \geq 0$ for all t, the curve is above the x-axis. It touches the x-axis only when $t = \pi/2$ and the point of intersection is $(0, 0)$

⑥ The curve intersects the y-axis whenever $\cos t = 0$. Since $-2\pi \leq t \leq 3\pi$, there are 5 values for t:

$$-\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$$

and the corresponding points are:

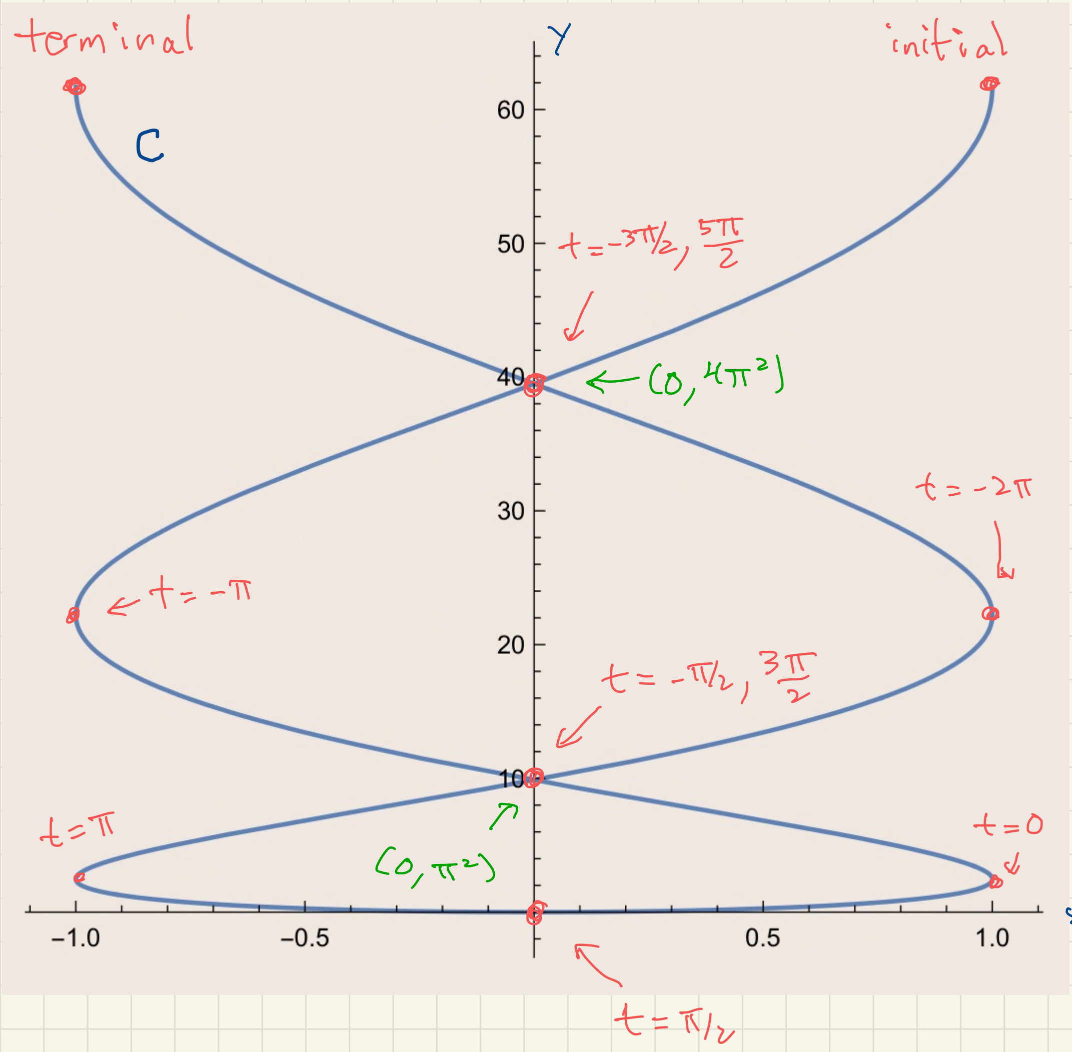
$$(0, 4\pi^2), (0, \pi^2), (0, 0), (0, \pi^2), (0, 4\pi^2)$$

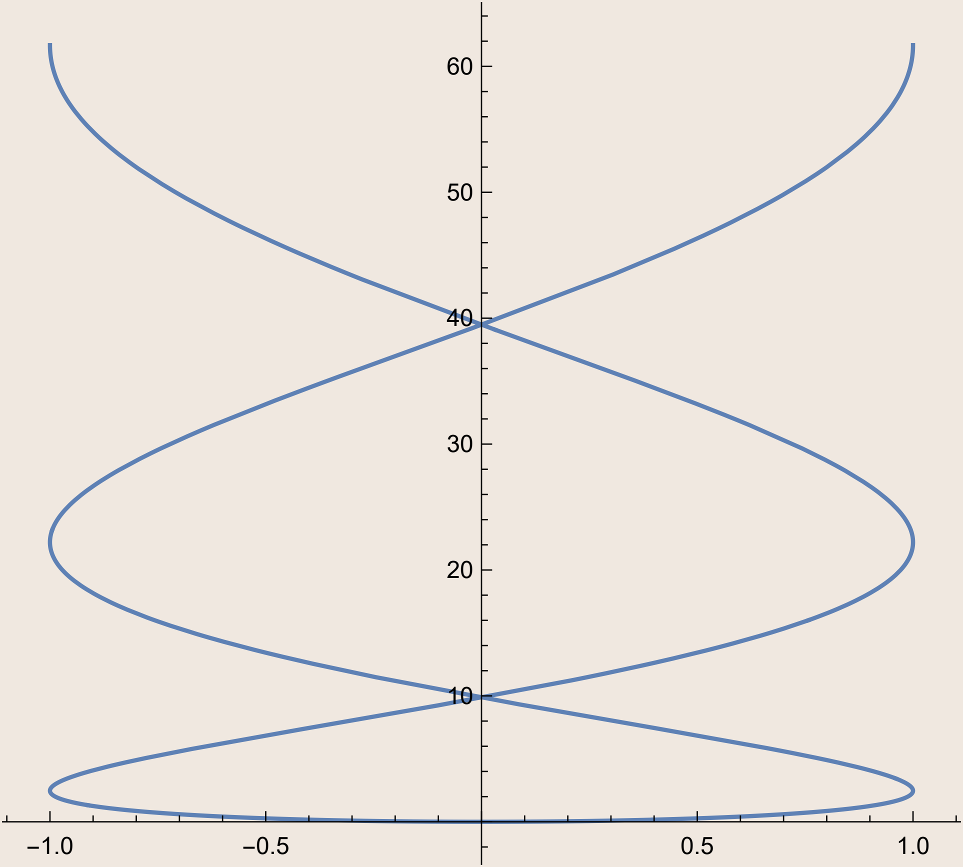
Observe that $(0, \pi^2)$ occurs for two values of t, as does $(0, 4\pi^2)$. These points of self intersection for C.

It turns out that the curve

$$C : \begin{cases} x = \cos(t) \\ y = \left(t - \frac{\pi}{2}\right)^2 \end{cases} \qquad -2\pi \le t \le 3\pi.$$

looks like:





60
50
40
30
20
10
−1.0
−0.5
0.5
1.0