The <u>cross product</u> of $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$ is

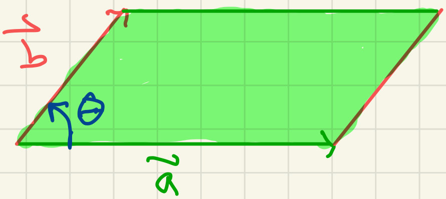
$$\vec{a} \times \vec{b} = \langle a_2 b_3 - a_3 b_2,\ a_3 b_1 - a_1 b_3,\ a_1 b_2 - a_2 b_1 \rangle$$

It is characterized by the following properties:

① $\vec{a} \perp \vec{a} \times \vec{b}$ and $\vec{b} \perp \vec{a} \times b$

② $|\vec{a} \times \vec{b}| = |\vec{a}|\,|\vec{b}|\,|\sin\theta|$ where θ is the angle between $\vec{a}$ and $\vec{b}$.

③ $(\vec{a}, \vec{b}, \vec{a} \times \vec{b})$ forms a right-hand system



← The area of the parallelogram determined by $\vec{a}$ and $\vec{b}$ equals $|\vec{a} \times \vec{b}| = |\vec{a}|\,|\vec{b}|\,|\sin\theta|$

It is very <u>important</u> to be able to calculate cross products quickly and accurately:

<u>quickly</u>: use 3×3 determinants : $\vec{a} = \langle a_1, a_2, a_3 \rangle$, $\vec{b} = \langle b_1, b_2, b_3 \rangle$

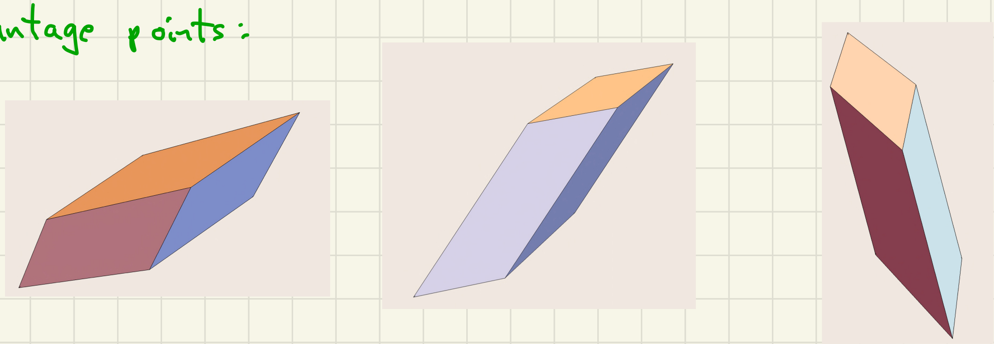
$$\vec{a} \times \vec{b} = \operatorname{Det} \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$$

(see discussion on page 855)

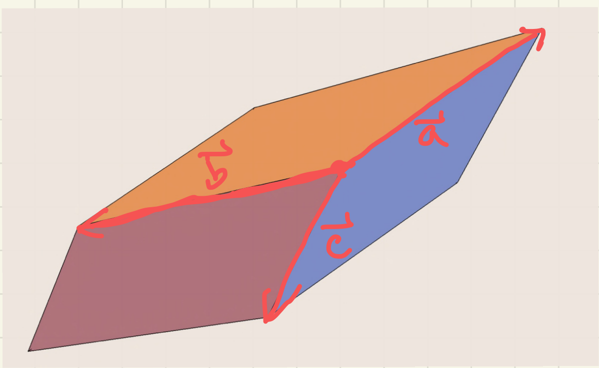
<u>accurately</u>: I always check the computation of $\vec{a} \times \vec{b}$ by calculating $(\vec{a} \times \vec{b}) \cdot \vec{a}$ and $(\vec{a} \times \vec{b}) \cdot \vec{b}$. If either is not equal to 0, then your computation of $\vec{a} \times \vec{b}$ is incorrect.

A parallelepiped is a 3D analogue of a parallelogram. There are 6 faces each of which is a parallelogram. Each face is parallel and congruent to its opposite face.

Here's a picture of one parallelepiped shown from three different vantage points:



Each parallepiped is determined by three vectors $\vec{a}, \vec{b}, \vec{c}$:



Fact The volume of parallelepiped determined by $\vec{a}, \vec{b}, \vec{c}$ is

$$\text{Volume} = \left| \vec{a} \cdot (\vec{b} \times \vec{c}) \right|.$$

The number $\vec{a} \cdot (\vec{b} \times \vec{c})$ is called the triple scalar product of $\vec{a}, \vec{b}, \vec{c}$. (Stewart, page 859)

Special Examples

- A rectangular box is a parallelepiped for which each of the 6 faces are rectangles.

example The parallelepiped determined by the vectors $\vec{a} = 2\vec{i}$, $\vec{b} = 3\vec{j}$, $\vec{c} = 4\vec{k}$ is a rectangular box because $\vec{a} \perp \vec{b}$, $\vec{a} \perp \vec{c}$, and $\vec{b} \perp \vec{c}$.

By the formula on the previous page, the volume of this box is

$$|\vec{a} \cdot (\vec{b} \times \vec{c})| = |2\vec{i} \cdot (3\vec{j} \times 4\vec{k})| = |2\vec{i} \cdot 12\vec{i}| = 24|\vec{i} \cdot \vec{i}| = 24$$

(note that $\vec{i} \cdot \vec{i} = |\vec{i}|^2 = 1$)

Of course this is just verifying the well-known fact that the volume of a rectangular box is width × length × height.

- A cube is a rectangular box where all 6 faces are squares. (Note that each square will have the same side length.)

Page 859:

11 Properties of the Cross Product If **a**, **b**, and **c** are vectors and c is a scalar, then

1. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
2. $(c\mathbf{a}) \times \mathbf{b} = c(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (c\mathbf{b})$
3. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
4. $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$
5. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$
6. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

(Each property can be verified by writing $\vec{a} = \langle a_1, a_2, a_3 \rangle$, $\vec{b} = \langle b_1, b_2, b_3 \rangle$, $\vec{c} = \langle c_1, c_2, c_3 \rangle$ and expanding LHS and RHS.)

<u>Consequence</u> Two vectors $\vec{a}$ and $\vec{b}$ are parallel if and only if $\vec{a} \times \vec{b} = \vec{0}$.

<u>explanation 1</u>: By property 1, $\vec{a} \times \vec{a} = -\vec{a} \times \vec{a}$, but the only vector $\vec{v}$ with $\vec{v} = -\vec{v}$ is $\vec{v} = \vec{0}$. If $\vec{b}$ is parallel to $\vec{a}$ then $\vec{b} = t\vec{a}$ for some scalar t, and, using property 2,

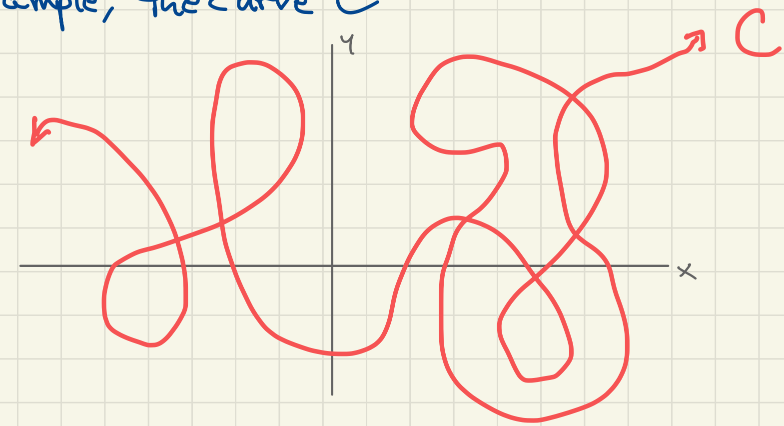
$$\vec{a} \times (t\vec{a}) = t(\vec{a} \times \vec{a}) = t\,\vec{0} = \vec{0}.$$

<u>explanation 2</u>: If $\vec{a}$ and $\vec{b}$ are parallel then the angle θ between $\vec{a}$ and $\vec{b}$ is $\theta = 0$ or $\theta = \pi$. In either case $\sin(\theta) = 0$. Then $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}||\sin\theta| = 0$. Then $\vec{a} \times \vec{b} = \vec{0}$ because $\vec{0}$ is only vector with length 0.

Next: Go back to section 10. Start with descriptions of curves by "parametric equations". (section 10.1, page 679)

In Calculus I and II, the geometry of curves that are graphs of functions was extensively studied. However there are lots of curves that don't have this form.

For example, the curve C



is not the graph of a function because it does not satisfy the VLP ($\equiv$ vertical line property). It can be described as the trace of an object moving in the xy-plane. This is the parametric equations perspective.

The curve will be described by a pair of functions

$$C: \begin{cases} x = f(t) \\ y = g(t) \end{cases}$$

where $(f(t), g(t))$ is the location of the object in the xy-plane at time t. The function $f(t)$ determines the horizontal position at time t, and $g(t)$ gives the vertical position.