Problem Does the series converge? If so find the sum?

(a) $$\sum_{n=0}^{\infty} \frac{5^n}{8^{2n+1}} = (a)$$

geometric series

$$1 + r + r^2 + r^3 + \dots = \frac{1}{1-r}$$

if $-1 < r < 1$

$$\sum_{n=1}^{\infty} r^{n-1} = \sum_{m=0}^{\infty} r^m$$

$$\frac{5^n}{8^{2n+1}} = \frac{1}{8} \cdot \frac{5^n}{8^{2n}}$$

$$= \frac{1}{8} \frac{5^n}{64^n} = \frac{1}{8}\left(\frac{5}{64}\right)^n$$

here $r = 5/64$

$$(a) = \sum_{n=0}^{\infty} \frac{1}{8}\left(\frac{5}{64}\right)^n = \frac{1}{8}\sum_{n=0}^{\infty}\left(\frac{5}{64}\right)^n$$

$$= \frac{1}{8} \frac{1}{1-\frac{5}{64}} \frac{64}{64}$$

Answer Series (a) converges with sum $8/59$.

$$= \frac{1}{8} \frac{64}{64-5} = \frac{8}{59}$$

(b) $$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} 3^{n-1}}{5^{n-1}} = \sum_{n=1}^{\infty} \left(-\frac{3}{5}\right)^{n-1}$$

converges to $5/8$

$$= 1 + \left(-\frac{3}{5}\right)^1 + \left(-\frac{3}{5}\right)^2 + \left(-\frac{3}{5}\right)^3 + \dots$$

here $r = -3/5$

$$= \frac{1}{1-(-3/5)} = \frac{1}{8/5} = 5/8$$

(b) $\displaystyle\sum_{n=1}^{\infty} \left(-\frac{3}{5}\right)^{n-1} = \frac{5}{8}$

converges

(c) $\displaystyle\sum_{n=4}^{\infty} \left(-\frac{3}{5}\right)^{n-1} = ??$

converges but sum may be different than (b)

geometric series

$$1 + r + r^2 + r^3 + \cdots = \frac{1}{1-r}$$

if $-1 < r < 1$

$$\sum_{n=1}^{\infty} r^{n-1} = \sum_{m=0}^{\infty} r^m$$

$$(c) = \left(-\frac{3}{5}\right)^4 + \left(-\frac{3}{5}\right)^5 + \left(-\frac{3}{5}\right)^6 + \cdots$$

$$(b) = 1 + \left(-\frac{3}{5}\right) + \left(-\frac{3}{5}\right)^2 + \left(-\frac{3}{5}\right)^3 + \underbrace{\left(-\frac{3}{5}\right)^4 + \cdots}_{(c)}$$

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \sum_{n=4}^{\infty} a_n$$

$$(c) = \underline{\left(\sum_{n=1}^{\infty} \left(-\frac{3}{5}\right)^{n-1}\right)} - \left(\underbrace{\left(-\frac{3}{5}\right)^0}_{n=1 \text{ term}} + \underbrace{\left(-\frac{3}{5}\right)^1}_{n=2 \text{ term}} + \underbrace{\left(-\frac{3}{5}\right)^2}_{n=3 \text{ term}}\right)$$

$$= \frac{5}{8} - \left(1 - \frac{3}{5} + \frac{9}{25}\right) = \ldots$$

(d) $\displaystyle\sum_{n=4}^{\infty} \left(-\frac{5}{3}\right)^{n-1}$

diverges, $r = -\frac{5}{3}$

$\left|-\frac{5}{3}\right| = \frac{5}{3} > 1$

geometric series for r (where r is a constant)

$$1 + r + r^2 + r^3 + \cdots = \sum_{n=1}^{\infty} r^{n-1}$$

This series diverges when $|r| \geq 1$, but converges when $|r| < 1$.

note: $|r| \geq 1$ means $r \geq 1$ or $r \leq -1$

$|r| < 1$ means $-1 < r < 1$

examples: $\sum_{n=1}^{\infty} 7^{n-1}$ diverges, $r = 7$

$\sum_{n=1}^{\infty} \frac{1}{7^{n-1}}$ converges, $r = 1/7$

$$\parallel$$

$$\sum_{n=1}^{\infty} (1/7)^{n-1}$$

example: $\sum \dfrac{7n^3 - 2n + 3}{12n^3 - n^2 + n + 1}$ converge or diverge?

Test For Divergence:

If $\lim_{n\to\infty} a_n \neq 0$ then $\sum a_n$ diverges.

$$a_n = \frac{7n^3 - 2n + 3}{12n^3 - n^2 + n + 1}$$

find limit as $n \to \infty$

$$\lim_{n\to\infty} \frac{7n^3 - 2n + 3}{12n^3 - n^2 + n + 1} \cdot \frac{1/n^3}{1/n^3} = \lim_{n\to\infty} \frac{7 - 2\cancel{\frac{1}{n^2}}^{\,0} + 3\cancel{\frac{1}{n^3}}^{\,0}}{12 - \cancel{\frac{1}{n}}_{\,0} + \cancel{\frac{1}{n^2}}_{\,0} + \cancel{\frac{1}{n^3}}_{\,0}}$$

$$= \frac{7 - 0 + 0}{12 - 0 + 0 + 0} = 7/12 = \lim_{n\to\infty} a_n$$

$$7/12 \neq 0$$

So
The original series diverges (using Test for Divergence).

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example $\sum \dfrac{7n^3 - 2n + 3}{12n^4 - n^2 + n + 1}$

$$\lim_{n\to\infty} \frac{7n^3 - 2n + 3}{12n^4 - n^2 + n + 1} = \lim_{n\to\infty} \frac{7\,1/n - 2\frac{1}{n^3} + 3\frac{1}{n^4}}{12 - 1/n^2 + 1/n^3 + 1/n^4} = 0$$

So in this example the Test for Divergence is inconclusive (doesn't give any information about whether the series converges or not). Later (in section 11.4) we'll be able to show it diverges.

$\cos(\pi) = -1$

$\cos(\pi/3) = 1/2$

$\sin(\pi/3) = \sqrt{3}/2$