$\sum_{n=1}^{\infty} \frac{1}{n^p}$

Infinite Series Review Sheet: Convergence Tests

TEST FOR DIVERGENCE. *If* $\lim_{n\to\infty} a_n \neq 0$ *then the series* $\sum_{n=1}^{\infty} a_n$ *diverges.*

LINEARITY. *If* c *is a constant and the series* $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *converge then* $\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} a + n + b_n = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$.

GEOMETRIC SERIES. *If* r *is a constant then* $\sum_{n=0}^{\infty} r^n$ *converges when* $|r| < 1$ *and diverges when* $|r| \geq 1$. *When* $|r| < 1$, *the sum of this series equals* $1/(1-r)$.

p-SERIES. *If* p *is a constant then* $\sum n = 1^{\infty} \frac{1}{n^p}$ *converges when* $p > 1$ *and diverges when* $p \leq 1$.

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positive series

INTEGRAL TEST. *Let* $f(x)$ *be a positive, continuous decreasing function for* $x \geq 1$ *and let* $a_n = f(n)$.
(a) *If the improper integral* $\int_1^{\infty} f(x)\,dx$ *converges then the series* $\sum_{n=1}^{\infty} a_n$ *converges.*
(b) *If the improper integral* $\int_1^{\infty} f(x)\,dx$ *diverges then the series* $\sum_{n=1}^{\infty} a_n$ *diverges.*

COMPARISON TEST. *Let* $\{a_n\}$ *and* $\{b_n\}$ *be sequences of positive numbers with* $a_n \leq b_n$ *for all positive integers* n.
(a) *If* $\sum_{n=1}^{\infty} b_n$ *converges then* $\sum_{n=1}^{\infty} a_n$ *converges.*
(b) *If* $\sum_{n=1}^{\infty} a_n$ *diverges then* $\sum_{n=1}^{\infty} b_n$ *diverges.*

LIMIT COMPARISON TEST. *Let* $\{a_n\}$ *and* $\{b_n\}$ *be sequences of positive numbers with* $\lim_{n\to\infty} \frac{a_n}{b_n} = c$.
(a) *If the series* $\sum_{n=1}^{\infty} b_n$ *converges and* $0 \leq c < \infty$ *then* $\sum_{n=1}^{\infty} a_n$ *converges.*
(b) *If the series* $\sum_{n=1}^{\infty} b_n$ *diverges and* $0 < c \leq \infty$ *then* $\sum_{n=1}^{\infty} a_n$ *diverges.*

RATIO TEST. *Suppose that* $a_n > 0$ *for all* n *and that* $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$.
(a) *If* $L < 1$ *then* $\sum_{n=1}^{\infty} a_n$ *converges.*
(b) *If* $L > 1$ *then* $\sum_{n=1}^{\infty} a_n$ *diverges. In fact, if* $L > 1$ *then* $\lim_{n\to\infty} a_n = \infty$.

ROOT TEST. *Suppose that* $a_n > 0$ *for all* n *and that* $\lim_{n\to\infty} \sqrt[n]{a_n} = L$.
(a) *If* $L < 1$ *then* $\sum_{n=1}^{\infty} a_n$ *converges.*
(b) *If* $L > 1$ *then* $\sum_{n=1}^{\infty} a_n$ *diverges .*

ALTERNATING SERIES TEST. *Let* $\{b_n\}$ *be a decreasing sequence with* $\lim_{n\to\infty} b_n = 0$ *and* $b_n > 0$. *Then the series* $\sum_{n=1}^{\infty} (-1)^n b_n$ *converges.*

ABSOLUTE CONVERGENCE TEST. *If* $\sum_{n=1}^{\infty} |a_n|$ *converges (that is, if* $\sum_{n=1}^{\infty} a_n$ *"converges absolutely") then* $\sum_{n=1}^{\infty} a_n$ *converges.*

ADDENDUM TO RATIO TEST. *If* $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$ *and* $L > 1$ *then* $\sum_{n=1}^{\infty} a_n$ *diverges.*

A series $\sum_{n=1}^{\infty} a_n$ is said to **converge absolutely** if the positive series $\sum_{n=1}^{\infty} |a_n|$ converges. Note that the Integral, Comparison, Limit Comparison, Root and Ratio Tests are all tests that apply only to positive series (or really to any series that has only finitely many negative terms). However they can be used to determine the absolute convergence of any series.

IMPORTANT BASIC PRINCIPLE: An infinite series $\sum_{n=M}^{\infty} a_n$ will converge if and only if the series $\sum_{n=L}^{\infty} a_n$ converges. This means that the value of the starting index for the series has no effect on whether it converges or diverges. So it is common to leave off the indexing entirely and just say that $\sum a_n$ converges or diverges. (The same comments apply in like manner for absolute convergence and conditional convergence.) However, if you want to determine the sum of a convergent series then that does depend on where the indexing starts. For example, $\sum_{n=0}^{\infty}(2/3)^n = 3$ but $\sum_{n=1}^{\infty}(2/3)^n = 2$.

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example $\displaystyle\sum_{n=1}^{\infty} \frac{3n-2}{5n^2+n}$ converge or diverge?

$\displaystyle\sum_{n=1}^{\infty} a_n$ $\uparrow$ series $\qquad a_n = \frac{3n-2}{5n^2+n}$

sequence of terms (list) $\{a_n\} = \{ \frac{1}{6}, \frac{2}{11}, \frac{7}{48}, \frac{5}{42}, \frac{1}{10}, \dots \}$

$\frac{1}{6} = a_1$, $\frac{2}{11} = a_2$, $\frac{7}{48} = a_3$, $\frac{5}{42} = a_4$, $\frac{1}{10} = a_5$

series $\frac{1}{6} + \frac{2}{11} + \frac{7}{48} + \frac{5}{42} + \frac{1}{10} + \dots$

sequence of partial sums $\{S_n\}$ ← too hard to calculate a pattern for these numbers. But

$S_4 = \frac{1}{6} + \frac{2}{11} + \frac{7}{48} + \frac{5}{42}$

Sum of series $= \lim_{n\to\infty} S_n =$ is this a finite number?

basic question

Always good to ask first:

$\{a_n\}$ ← Does $\lim_{n\to\infty} a_n$ exist? what is it?

$$\lim_{n\to\infty} \frac{3n-2}{5n^2+n} \cdot \frac{1/n^2}{1/n^2} = \lim_{n\to\infty} \frac{3\cancel{\frac{1}{n}} - 2\cancel{\frac{1}{n^2}}}{5 + \cancel{\frac{1}{n}}} = \frac{0}{5} = 0$$

here, Test for Divergence says nothing.

$$\sum \frac{3n-2}{5n^2+n} = \sum a_n$$

Use comparison test

In this example Look for p-series

Compare to some other series $\sum b_n$ which we know either converges or diverges.

$$\overset{a_n}{\overset{\shortparallel}{\frac{3n-2}{5n^2+n}}} < \frac{3n}{5n^2+n} < \frac{3n}{5n^2} = \frac{3}{5}\,\frac{1}{n}$$

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{3}{5}\,\frac{1}{n} = \frac{3}{5} \sum_{n=1}^{\infty} \frac{1}{n}$$

p-series $p=1$ diverges

inequality in wrong direction !

So try limit comparison instead (with same b_n).

Use limit comparison test

Take $\sum b_n = \sum \frac{3}{5}\frac{1}{n}$. ← diverges

means $\sum a_n$ converges if and only if $\sum b_n$ converges

$$\frac{a_n}{b_n} = \frac{3n-2}{5n^2+n} \cdot \frac{5n}{3} = \frac{15n^2-10n}{15n^2+3n} \xrightarrow[n\to\infty]{} 1 = c$$

Since c is a finite number $\sum a_n$ converges if and only if $\sum b_n$ converges, and $\sum \frac{3}{5}\frac{1}{n}$ diverges

So $\sum a_n$ diverges by limit comparison test.

example $\sum \frac{3n-2}{5n^3+n^2}$ ← apply limit comparison with $\sum b_n = \sum \frac{1}{n^2}$

Here you could use comparison test

$$\frac{3n-2}{5n^3+n^2} < \frac{3n}{5n^3} = \frac{3}{5}\frac{1}{n^2} \, . \quad \sum \frac{3}{5}\frac{1}{n^2} \text{ converges, } p=2$$

So $\sum \frac{3n-2}{5n^3+n^2}$ converges.

example $\sum_{n=1}^{\infty} (-1)^n \frac{3n-2}{5n^2+n}$ converge?

not a positive series (some positive and some negative terms)

Try the alternating series test.

must verify these

$\left\{ \frac{3n-2}{5n^2+n} \right\}$ must be a decreasing with limit 0.

means: as n increases $\frac{3n-2}{5n^2+n}$ decreases

Consider $f(x) = \frac{3x-2}{5x^2+x}$ decreasing for $x>1$?

$$f'(x) = \frac{-15x^2+20x+2}{(5x^2+x)^2}$$

This is negative when $x \geq 2$. so this is true.

example

$$\sum \frac{5n^5 + n^3 - n + 1}{10n^9 + 2} = \sum a_n$$

$$b_n = \frac{n^5}{n^9} = \frac{1}{n^4}$$

use limit comparison

$\sum \frac{1}{n^4}$ converges $\Longrightarrow$ $\sum a_n$ converges

To verify the use of limit comparison test you must write out $\frac{a_n}{b_n}$ and compute its limit as $n \to \infty$ to find the value of c. . . .

from Webwork 8:

#10

Each of the following statements is an attempt to show that a given series is convergent or divergent using the Comparison Test (NOT the Limit Comparison Test.) For each statement, enter C (for "correct") if the argument is valid, or enter I (for "incorrect") if any part of the argument is flawed. (Note: if the conclusion is true but the argument that led to it was wrong, you must enter I.)

___1. For all $n > 2$, $\frac{\ln(n)}{n^2} > \frac{1}{n^2}$, and the series $\sum \frac{1}{n^2}$ converges, so by the Comparison Test, the series $\sum \frac{\ln(n)}{n^2}$ converges. I

___2. For all $n > 2$, $\frac{n}{n^3-9} < \frac{2}{n^2}$, and the series $2\sum \frac{1}{n^2}$ converges, so by the Comparison Test, the series $\sum \frac{n}{n^3-9}$ converges. C

___3. For all $n > 1$, $\frac{\ln(n)}{n^2} < \frac{1}{n^{1.5}}$, and the series $\sum \frac{1}{n^{1.5}}$ converges, so by the Comparison Test, the series $\sum \frac{\ln(n)}{n^2}$ converges. C

___4. For all $n > 1$, $\frac{\arctan(n)}{n^3} < \frac{\pi}{2n^3}$, and the series $\frac{\pi}{2}\sum \frac{1}{n^3}$ converges, so by the Comparison Test, the series $\sum \frac{\arctan(n)}{n^3}$ converges. C

___5. For all $n > 2$, $\frac{1}{n^2-3} < \frac{1}{n^2}$, and the series $\sum \frac{1}{n^2}$ converges, so by the Comparison Test, the series $\sum \frac{1}{n^2-3}$ converges. I

___6. For all $n > 2$, $\frac{\ln(n)}{n} > \frac{1}{n}$, and the series $\sum \frac{1}{n}$ diverges, so by the Comparison Test, the series $\sum \frac{\ln(n)}{n}$ diverges. C

False: The inequality is going in wrong direction

False: $\frac{1}{n^2-3} > \frac{1}{n^2}$

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#20. Use the ratio test to determine whether $\sum_{n=20}^{\infty} \frac{n(-2)^n}{n!}$ converges or diverges.

(a) Find the ratio of successive terms. Write your answer as a fully simplified fraction. For $n \geq 20$,

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \underline{\qquad\qquad}$$

see next page

(b) Evaluate the limit in the previous part. Enter ∞ as *infinity* and $-\infty$ as *-infinity*. If the limit does not exist, enter *DNE*.

$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \underline{\qquad\qquad}$

(c) By the ratio test, does the series converge, diverge, or is the test inconclusive?

- Choose
- Converges
- Diverges
- Inconclusive

#17. Consider the series $\sum_{n=1}^{\infty} \frac{10^n}{(n+1)6^{2n+1}}$. Evaluate the the following limit. If it is infinite, type "infinity" or "inf". If it does not exist, type "DNE".

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$$

Answer: $L =$ 10/36

What can you say about the series using the Ratio Test? Answer "Convergent", "Divergent", or "Inconclusive".
Answer:

- choose one
- Convergent
- Divergent
- Inconclusive

Determine whether the series is *absolutely convergent, conditionally convergent,* or *divergent.* Answer "Absolutely Convergent", "Conditionally Convergent", or "Divergent".
Answer:

- choose one
- Absolutely Convergent
- Conditionally Convergent
- Divergent

example

$$\sum_{n=20}^{\infty} \frac{n(-2)^n}{n!}$$

use ratio test

$a_n = \frac{n(-2)^n}{n!}$, $|a_n| = \frac{n2^n}{n!}$

$|a_{n+1}| = \frac{(n+1)2^{n+1}}{(n+1)!}$ $\qquad \frac{n!}{(n+1)!} = \frac{1}{n+1}$

$$\frac{|a_{n+1}|}{|a_n|} = \frac{(n+1)2^{n+1}}{(n+1)!} \frac{n!}{n2^n} = \frac{n+1}{n}\frac{2}{n+1} = \frac{2}{n}$$

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{2}{n} \doteq 0 = L$$

Ratio says $\sum_{n=20}^{\infty} |a_n|$ converges b/c $L<1$

$= |a_{20}| + |a_{21}| + |a_{22}| + \cdots$

So $\sum_{n=20}^{\infty} a_n$ converges absolutely

note: $|a+b| \neq |a|+|b|$

example $|1+(-1)| = 0$

but $|1|+|-1| = 2$

#14.

Use the limit comparison test to determine whether $\sum_{n=13}^{\infty} a_n = \sum_{n=13}^{\infty} \frac{9n^3 - 3n^2 + 13}{5 + 3n^4}$ converges or diverges.

(a) Choose a series $\sum_{n=13}^{\infty} b_n$ with terms of the form $b_n = \frac{1}{n^p}$ and apply the limit comparison test. Write your answer as a fully simplified fraction. For $n \geq 13$,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \_\_\_\_\_\_$$

Take $b_n = \frac{n^3}{n^4} = \frac{1}{n}$

(So $p=1$)

(b) Evaluate the limit in the previous part. Enter ∞ as *infinity* and $-\infty$ as *-infinity*. If the limit does not exist, enter *DNE*.

$\lim_{n\to\infty} \frac{a_n}{b_n} = \_\_\_\_\_\_$

(c) By the limit comparison test, does the series converge, diverge, or is the test inconclusive?

- Choose
- Converges
- Diverges ✓
- Inconclusive

#9.

Use the Integral Test to determine whether the infinite series is convergent.

$$\sum_{n=16}^{\infty} \frac{n^2}{(n^3+9)^{\frac{7}{2}}}$$

To perform the integral test, one should calculate the improper integral

$\int_{16}^{\infty} \_\_\_\_\_\_ \, dx = \_\_\_\_\_\_$

Enter **inf** for ∞, **-inf** for $-\infty$, and **DNE** if the limit does not exist.

By the Integral Test,

the infinite series $\sum_{n=16}^{\infty} \frac{n^2}{(n^3+9)^{\frac{7}{2}}}$

- A. converges ✓
- B. diverges

$$\int \frac{x^2}{(x^3+9)^{7/2}}\,dx = \frac{1}{3}\int u^{-7/2}\,du$$

$$= \ldots$$

$\left(\begin{array}{l} u = x^3+9 \\ du = 3x^2\,dx \end{array} \right.$

#19.

Test the series for convergence or divergence.

$$\sum_{n=1}^{\infty} \frac{n!}{111^n}.$$

Use the

- Select
- Ratio Test ✓
- Root Test

and evaluate:

$\lim_{n\to\infty} \_\_\_\_\_\_ = \underline{\infty}$. (Note: Use INF for an infinite limit.)

Since the limit is

- Select
- finite
- greater than 1 ✓
- equal to 1
- less than 1
- greater than 0
- equal to 0

,

- Select
- the series diverges ✓
- the series converges conditionally
- the series converges absolutely
- we know nothing

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Test your understanding of Convergence Tests to answer the Basic Question $\boxed{BQ}$ with these..

Exercises from Stewart : Some Easy – Some Not So Easy

should be able to apply indicated tests.

11.7 EXERCISES

1–38 Test the series for convergence or divergence.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2-1}{n^3+1}$ diverges LCT

2. $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{n^3+1}$ converges CT

3. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^2-1}{n^3+1}$ AST

4. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^2-1}{n^2+1}$ DT

5. $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{n^2}$ DT

6. $\displaystyle\sum_{n=1}^{\infty} \frac{n^{2n}}{(1+n)^{3n}}$ Root

7. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$ IT

8. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^4}{4^n}$ Abs

9. $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n}}{(2n)!}$ Ratio

10. $\displaystyle\sum_{n=1}^{\infty} n^2 e^{-n^3}$ IT

11. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n^3}+\frac{1}{3^n}\right)$ PS + GS

12. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k\sqrt{k^2+1}}$ CT

13. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n n^2}{n!}$ Ratio

14. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin 2n}{1+2^n}$ Abs + CT

15. $\displaystyle\sum_{k=1}^{\infty} \frac{2^{k-1}3^{k+1}}{k^k}$ Ratio

16. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n^4+1}}{n^3+n}$ LCT

17. $\displaystyle\sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdot \cdots \cdot (2n-1)}{2\cdot 5\cdot 8\cdot \cdots \cdot (3n-1)}$ Ratio

18. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}-1}$ AST

19. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{\sqrt{n}}$ AST

20. $\displaystyle\sum_{k=1}^{\infty} \frac{\sqrt[3]{k}-1}{k(\sqrt{k}+1)}$ LCT

21. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \cos(1/n^2)$ DT

22. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{2+\sin k}$ DT

23. $\displaystyle\sum_{n=1}^{\infty} \tan(1/n)$ CT

24. $\displaystyle\sum_{n=1}^{\infty} n\sin(1/n)$ DT

25. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{e^{n^2}}$ Ratio

26. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+1}{5^n}$ LCT

27. $\displaystyle\sum_{k=1}^{\infty} \frac{k\ln k}{(k+1)^3}$ CT

28. $\displaystyle\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2}$ CT

29. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\cosh n}$ AST

30. $\displaystyle\sum_{j=1}^{\infty} (-1)^j \frac{\sqrt{j}}{j+5}$ AST

31. $\displaystyle\sum_{k=1}^{\infty} \frac{5^k}{3^k+4^k}$ DT

32. $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^n}{n^{4n}}$ root

33. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$ Root

34. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+n\cos^2 n}$ CT

35. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$ LCT

36. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(\ln n)^{\ln n}}$ LCT

37. $\displaystyle\sum_{n=1}^{\infty} (\sqrt[n]{2}-1)^n$ root

38. $\displaystyle\sum_{n=1}^{\infty} (\sqrt[n]{2}-1)$ LCT

NOTE : Many of these can be solved using more than one method.