Infinite Series: Attempt to add an infinite list of numbers.

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + \cdots$$

Summation Notation

$$\sum_{n=1}^{\infty} (n-3)^2 5^{n+1} = \infty$$

$a_n = (n-3)^2 5^{n+1}$

$a_1 = 4 \cdot 5^2$

$a_2 = 1 \cdot 5^3$

$a_3 = 0$

$a_4 = 1 \cdot 5^5$

$a_5 = 4 \cdot 5^6$

$$= \underset{a_1}{4 \cdot 5^2} + \underset{a_2}{1 \cdot 5^3} + \underset{a_3}{0} + \underset{a_4}{1 \cdot 5^5} + \cdots$$

$$\sum_{n=1}^{\infty} 0 = 0 + 0 + 0 + 0 + \cdots = 0$$

attempt to add is successful.

Reindexing a series

$$\sum_{n=3}^{\infty} a_{n-2} = a_1 + a_2 + a_3 + a_4 + \cdots$$

$$= \sum_{n=1}^{\infty} a_n$$

will give more examples with geometric series ....

Geometric Series

r = fixed number

$$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + r^4 + \cdots$$

idea: $(1 + r + r^2 + r^3 + r^4)(1 - r)$

$$= (1 + \cancel{r} + \cancel{r^2} + \cancel{r^3} + \cancel{r^4}) - (\cancel{r} + \cancel{r^2} + \cancel{r^3} + \cancel{r^4} + r^5)$$

$$= 1 - r^5$$

$$S_n = 1 + r + \cdots + r^n = \frac{1 - r^{n+1}}{1-r} = \frac{1}{1-r} - \cancel{\frac{r^{n+1}}{1-r}} \to 0$$

when $|r|<1$

Take limit as $n \to \infty$, get $\frac{1}{1-r}$

$$\boxed{\sum_{n=0}^{\infty} r^n = \frac{1}{1-r} \quad \text{when } -1 < r < 1}$$

What are 3 associated objects with $\sum_{n=0}^{\infty} r^n$ / $\sum_{n=0}^{\infty} a_n$

- $\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + \cdots$ — series itself
- list of numbers to be added — sequence of terms
  $\{1, r, r^2, r^3, r^4, \ldots\}$
  $= \{a_0, a_1, a_2, \ldots\}$
- $s_n = a_0 + a_1 + a_2 + \cdots + a_n$ — sequence of partial sum

If $\sum_{n=0}^{\infty} a_n$ has a sum then it equals $\lim_{n\to\infty} s_n$

recap: $\boxed{\dfrac{1}{1-r} = \sum_{n=0}^{\infty} r^n = 1 + r + r^2 + \cdots \quad -1<r<1}$

examples with geometric series

① $\displaystyle\sum_{n=0}^{\infty} \frac{5^n}{7^n} = \sum_{n=0}^{\infty} \left(\frac{5}{7}\right)^n = \sum_{n=0}^{\infty} r^n, \quad r = 5/7$

$= 1 + \frac{5}{7} + \left(\frac{5}{7}\right)^2 + \left(\frac{5}{7}\right)^3 + \cdots$

Since $-1 < \frac{5}{7} < 1$, so $\displaystyle\sum_{n=0}^{\infty} \frac{5^n}{7^n} = \frac{1}{1-\frac{5}{7}} = \frac{7}{2}$

② $\displaystyle\sum_{n=1}^{\infty} \left(\frac{5}{7}\right)^n = \underset{-1}{1+}\left(\frac{5}{7}\right) + \left(\frac{5}{7}\right)^2 + \left(\frac{5}{7}\right)^3 + \cdots = \frac{7}{2} - 1 = \frac{5}{2}$

③ $\displaystyle\sum_{n=4}^{\infty} \left(\frac{5}{7}\right)^{n-4} = \sum_{n=0}^{\infty} \left(\frac{5}{7}\right)^n = \frac{7}{2}$ re-indexing

$\parallel \qquad \parallel$

$1 + \frac{5}{7} + \left(\frac{5}{7}\right)^2 + \left(\frac{5}{7}\right)^3 + \cdots$

④ $\displaystyle\sum_{n=4}^{\infty} \left(\frac{5}{7}\right)^n = \left(\frac{5}{7}\right)^4 + \left(\frac{5}{7}\right)^5 + \left(\frac{5}{7}\right)^6 + \cdots$

$1 + \left(\frac{5}{7}\right) + \left(\frac{5}{7}\right)^2 + \left(\frac{5}{7}\right)^3 - \left(1 + \frac{5}{7} + \left(\frac{5}{7}\right)^2 + \left(\frac{5}{7}\right)^3\right)$

$= \frac{7}{2} - \left(1 + \frac{5}{7} + \left(\frac{5}{7}\right)^2 + \left(\frac{5}{7}\right)^3\right)$

⑤ $\displaystyle\sum_{n=1}^{\infty} \frac{5^{n+1}}{7^{n+3}} = \sum_{n=1}^{\infty} \frac{5 \cdot 5^n}{7^3 \cdot 7^n} = \frac{5}{7^3} \sum_{n=1}^{\infty} \left(\frac{5}{7}\right)^n$

$\frac{5}{7^3}\left(\underset{1-1}{}\ \frac{5}{7} + \left(\frac{5}{7}\right)^2 + \left(\frac{5}{7}\right)^3 + \cdots\right) = \frac{5}{7^3}\left(\frac{7}{2} - 1\right)$

example

Does $\sum_{n=0}^{\infty} \frac{1}{5^n + 7^n}$ converge? Yes

Not a geometric series!! But we can relate it to one

$$\frac{1}{5^n+7^n} < \frac{1}{7^n} \quad \text{b/c } 5^n+7^n > 7^n$$

And $\sum_{n=0}^{\infty} \frac{1}{7^n} = \sum_{n=0}^{\infty} \left(\frac{1}{7}\right)^n$ ← geometric series for $r = 1/7$

$$= \frac{1}{1-\frac{1}{7}} = \frac{7}{6}$$

Therefore, Comparison Test $\Rightarrow \sum_{n=0}^{\infty} \frac{1}{5^n+7^n}$ converge, and its sum is less than 7/6

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sum of two geometric series

example $\sum_{n=0}^{\infty} \left(\frac{1}{5^n} + \frac{1}{7^n}\right) = \sum_{n=0}^{\infty} \frac{1}{5^n} + \sum_{n=0}^{\infty} \frac{1}{7^n}$

$r = \frac{1}{5}$ $\qquad$ $r = \frac{1}{7}$

$$= \frac{1}{1-\frac{1}{5}} + \frac{1}{1-1/7}$$

$$= \frac{5}{4} + \frac{7}{6} = \frac{29}{12}$$

So the series Converges! And we have also answered the secondary question.