Power Series centered at $x=a$:

$a_n = c_n(x-a)^n$

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = \sum_{n=0}^{\infty} a_n$$

has an associated <u>radius of convergence</u> R with the property that the series

- converges when $|x-a| < R$, and
- diverges when $|x-a| > R$.

This means that the interval of convergence I of the power series is one of:

$$(a-R, a+R),\ [a-R, a+R),\ (a-R, a+R],\ [a-R, a+R]$$

open interval — $(a-R, a+R)$; half-open interval — $[a-R, a+R)$, $(a-R, a+R]$; closed interval — $[a-R, a+R]$

Notice that each of these possible intervals has center (that is, midpoint) at $x=a$.

Strategy for determining interval of convergence:

(i) Use Ratio Test to find R.

(ii) Use other tests to determine convergence at the two endpoints: $x = a-R$, $x = a+R$

(iii) In the cases where $R=0$ or $R=\infty$, step (2) will be unnecessary.

Example $\displaystyle\sum_{n=1}^{\infty} \frac{(x-5)^n}{n^2\, 3^n}$. Find R and I

$$= \sum_{n=1}^{\infty} c_n (x-a)^n$$

center $a = 5$

$$\frac{n^2}{n^2+2n+1} \cdot \frac{1/n^2}{1/n^2} = \frac{1}{1+2\frac{1}{n}+\frac{1}{n^2}}$$

$c_n = \dfrac{1}{n^2 3}$

$$\frac{|a_{n+1}|}{|a_n|} = \frac{|x-5|^{n+1}}{(n+1)^2\, 3^{n+1}} \cdot \frac{n^2\, 3^n}{|x-5|^n} = \frac{1}{3} \underbrace{\frac{n^2}{(n+1)^2}}_{\xrightarrow{n\to\infty} 1} |x-5|$$

what does ratio test tell us now?

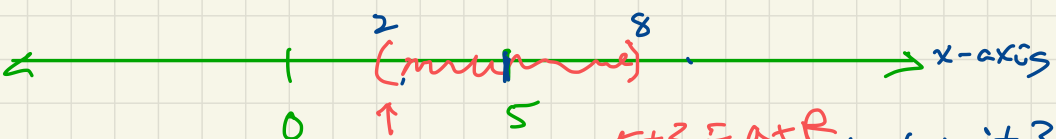
$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \frac{1}{3}|x-5| = L$$

Series converges when $L < 1$, diverges $L > 1$.

$\Updownarrow$ $\frac{1}{3}|x-5| < 1$ $\Updownarrow$ $|x-5| < 3$

$\Updownarrow$ $\frac{1}{3}|x-5| > 1$

$|x-5| > 1$



$a - R = 5-3$

$5+3 = a+R$

which is it?

So $R = 3$. and $I = (2,8), [2,8), (2,8], [2,8]$

$$\sum_{n=1}^{\infty} \frac{(x-5)^n}{n^2\, 3^n}.$$

We've determined so far, the power series for values of x in an interval with radius 3 centered 5. There are 4 possibilities

$(2,8) = \{x \mid 2 < x < 8\}$, $[2,8) = \{x \mid 2 \le x < 8\}$,
$(2,8] = \{x \mid 2 < x \le 8\}$, $[2,8] = \{x \mid 2 \le x \le 8\}$.

$\underline{\underline{x=2}}$ $$\sum_{n=1}^{\infty} \frac{(-3)^n}{n^2\, 3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$ converges absolutely.

converge? Yes absolutely

here

$$\sum \left| \frac{(-1)^n}{n^2} \right| = \sum \frac{1}{n^2}$$

$\underline{\underline{x=8}}$ $$\sum_{n=1}^{\infty} \frac{\cancel{3^n}}{n^2\, \cancel{3^n}} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$ converges. (p-series $p=2$)

<u>Final Answer</u> $I = [2,8]$

$\sum \frac{1}{n^2}$ converges
$\sum \frac{1}{n}$ diverges
p-series

$$\sum \frac{1}{n^p} = \begin{cases} \text{diverges} & p \le 1 \\ \text{converges} & p > 1 \end{cases}$$

If $\sum |a_n|$ converges then $\sum a_n$ converges.
(we say $\sum a_n$ <u>converges absolutely</u>)

positive series $\rightarrow$ use Ratio Test

example

$$\sum_{n=1}^{\infty} \frac{5^n (2n)!}{4 \cdot 7 \cdot 10 \cdots (3n+1)} = \sum a_n$$

$$\frac{a_{n+1}}{a_n} = \frac{5^{n+1} (2n+2)!}{4 \cdot 7 \cdot 10 \cdots (3n+4)} \cdot \frac{4 \cdot 7 \cdot 10 \cdots (3n+1)}{5^n (2n)!}$$

$$= \frac{5 (2n+2)(2n+1)}{3n+4} \xrightarrow[n \to \infty]{} \infty = L \quad \text{divergent}$$

$$\frac{5(4n^2+6n+2)}{3n+4}$$

$$\frac{\cancel{4} \cdot \cancel{7} \cdot \cancel{10} \cdots \cancel{(3n+1)}}{\cancel{4} \cdot \cancel{7} \cdot \cancel{10} \cdots (3n+1)(3n+4)} = \frac{1}{3n+4}$$

note:

$(n+1)! = (n+1) \cdot n!$ $\Rightarrow$

$$\frac{(2n+2)!}{(2n)!} = \frac{(2n+2)(2n+1)!}{(2n)!} = \frac{(2n+2)(2n+1)\cancel{(2n)!}}{\cancel{(2n)!}}$$

$$= (2n+2)(2n+1)$$

Webwork 9 problem #10 — some comments

va.pg

Select the FIRST correct reason why the given series converges.

A. ~~Convergent geometric series~~

B. ~~Convergent p series~~

C. Comparison (or Limit Comparison) with a geometric or p series

→ D. Alternating Series Test

E. None of the above

$\frac{5^n}{7^{2n}} = \left(\frac{5}{7^2}\right)^n$

___1. $\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{n+2}$ ← alt. series

A ___2. $\sum_{n=1}^{\infty} \frac{4(5)^n}{7^{2n}}$ ← $4 \sum \left(\frac{5}{49}\right)^n$

___3. $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{\ln(5n)}$ ← alt. series

n	$\cos(n\pi)$
0	1
1	−1
2	1
3	−1

___4. $\sum_{n=1}^{\infty} \frac{(-1)^n}{4n+5^n}$

B ___5. $\sum_{n=1}^{\infty} \frac{\cancel{(-1)^n} \ln(e^n)}{n^5 \cancel{\cos(n\pi)}} = \sum_{n=1}^{\infty} \frac{1}{n^4}$

___6. $\sum_{n=1}^{\infty} \frac{n^2+\sqrt{n}}{n^4-2}$

note:
$\boxed{\cos(n\pi) = (-1)^n}$

example

$$\sum (-1)^n \frac{\sqrt{n}}{2n+3} \quad \text{converge?}$$

Ultimate answer:
Yes it converges conditionally

Not a positive series — it is an alternating series

First question Does it converge absolutely?

i.e. — Does $\sum \left| (-1)^n \frac{\sqrt{n}}{2n+3} \right| = \sum \frac{\sqrt{n}}{2n+3} = \sum a_n$ converge?

Take $b_n = \frac{n^{1/2}}{n} = \frac{1}{n^{1/2}}$

$\sum b_n = \sum \frac{1}{n^{1/2}}$ ← diverges, p-series, $p = 1/2 \leq 1$

$$\frac{a_n}{b_n} = \frac{n^{1/2}}{2n+3} \cdot \frac{n^{1/2}}{1} = \frac{n}{2n+3} \xrightarrow[n\to\infty]{} \frac{1}{2} = c$$

$\Rightarrow \sum \frac{\sqrt{n}}{2n+3}$ diverges

So Given series does not converge absolutely.

Second Question : Can we use AST? Yes.

That will show that the series converges!

but must check conditions for AST.

decreasing sequence .... (no time for details now)

<u>Example</u> Does the Alternating Series Test apply to $\sum_{n=1}^{\infty} (-1)^n \frac{3n}{7n^2-1}$??