Some Review Problems for Exam 4—with some answers

Math 2433

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1. A line ℓ passes through the points $(1, -2, 3)$ and $(-2, 2, 5)$.
(a) Give scalar parametric equations for ℓ.
(b) Determine all of the points where ℓ crosses the three coordinate planes.
answer:
(a) $\ell : x = 3t + 1, y = -4t - 2, z = -2t + 3$
(b) They are $(0, -2/3, 11/3)$, $(-1/2, 0, 4)$ and $(11/2, -8, 0)$.

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2. A plane contains the line ℓ of the previous problem and passes through $(1, 3, -2)$. Determine an equation for this plane.

answer:
$2x + y + z = 3$

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3. Consider the plane $\mathcal{P}$ with equation $3x - 4y + 7z = -4$ and the line ℓ with scalar parametrization $\ell : \{x = 7t + 1, y = 14t, z = 5t - 1\}$.
(a) Find a normal vector for $\mathcal{P}$.
(b) Describe all possible normal vectors for $\mathcal{P}$.
(c) Give a vector parametrization for the line ℓ.
(d) Do the line and plane intersect? If so find all points of intersection.

answer:
(a) $\langle 3, -4, 7 \rangle$
(b) $\langle 3k, -4k, 7k \rangle$ where $k \neq 0$
(c) $\mathbf{r}(t) = \langle 7t + 1, .14t, 5t - 1 \rangle$
(d) Yes, ℓ is contained in $\mathcal{P}$.

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4. Let $\ell : x = -3t + 1, y = 2t - 3, z = t - 4$ be a scalar parametrization of the line ℓ.
(a) Does ℓ pass through the point $(4, -5, -4)$? Justify your answer.
(b) Give a parametrization of the line through the origin parallel to ℓ.
(c) Give an equation for the plane through the origin perpendicular to ℓ.
(d) Give an equation for the plane through the origin which contains ℓ.

answer: (a) No.
(b) $x = -3t, y = 2t, z = t$
(c) $-3x + 2y + z = 0$
(d) $5x + 11y - 7z = 0$

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5. Let ℓ be the line $\ell : \mathbf{r}(t) = \langle t + 2, t - 3, 2t + 1 \rangle$. Let $\mathcal{P}$ be the plane which contains ℓ and the point $P = (-1, 1, 1)$.
(a) Find an equation for the plane $\mathcal{P}$.
(b) Describe the line of intersection of $\mathcal{P}$ with the xy-plane.

answer: (a) $x + y + 2z - 2 = 0$
(b) The line has scalar parametrization $x = t, y = -t + 2, z = 0$

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6. Give an equation for the plane passing through the point $(-3, -1, 2)$ and parallel to the plane with equation $5x + 6y + z - 2 = 0$.

answer: $5x + 6y + z + 19 = 0$

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7. A plane contains the y-axis and $(1, -2, 3)$. Find an equation for it.

answer: $3x - z = 0$

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8. Show that the two planes $x + 3y - z = 1$ and $-2x - y + 3z = 0$ intersect in a line. Find the cosine of the angle between the two planes.

answer: Normal vectors for the two planes are $\langle 1, 3, -1\rangle$ and $\langle -2, -1, 3\rangle$ which are not parallel to each other (so the planes are not parallel and must intersect in a line). The cosine of the angle between the planes is

$$\frac{\langle 1, 3, -1\rangle \cdot \langle -2, -1, 3\rangle}{|\langle 1, 3, -1\rangle||\langle -2, -1, 3\rangle|} = -\frac{8}{\sqrt{154}}$$

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9. How many points are there in the intersection of the two planes $6x - 3y + 9z + 12 = 0$ and $-8x + 4y - 12z - 16 = 0$? Explain.

answer: Infinitely many (the two planes are identical).

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10. Let $\mathcal{P}$ be the plane with equation $2x - 5y + z + 3 = 0$ and let Q be the point $(4, -1, -1)$.
(a) Find an equation for the plane which is parallel to $\mathcal{P}$ and contains Q.
(b) Give a direction vector for the line which is perpendicular to $\mathcal{P}$ and passes through Q, and then determine a parametrization for this line.
(c) Use your answer to (b) to find the point on $\mathcal{P}$ closest to Q (that is, the foot of the perpendicular from Q to $\mathcal{P}$).

answer: (a) $2x - 5y + z + 12 = 0$
(b) $x = 2t + 4, y = -5t - 1, z = t - 1$
(c) $(12/5, 3, -9/5)$

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11. Let ℓ be the line of intersection of the two planes with equations $x + y - z = 2$ and $3x - 4y + 5z = 6$.
(a) Give a scalar parametrization for ℓ.
(b) Give a vector parametrization for ℓ.
(c) Does ℓ pass through the origin?
(d) Find the point of intersection of ℓ with the yz-coordinate plane.

answer: (a) $x = t + 2, y = -8t, z = -7t$

(b) $\mathbf{r}(t) = \langle 1, -8, -7\rangle t + \langle 2, 0, 0\rangle$
(c) No. In fact the origin $O = (0, 0, 0)$ does not lie on either of the two planes.
(d) The yz-plane has equation $x = 0$. Setting x to 0 in $x + y - z = 2$ and $3x - 4y + 5z = 6$ gives the equations $y - z = 2$ and $-4y + 5z = 6$, and solving these gives $(x, y, z) = (0, 16, 14)$

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12. Let ℓ_1 and ℓ_2 be lines with parametrizations $\ell_1 : x = -t + 1, y = 2t - 1, z = t + 3$ and $\ell_2 : x = 2t, y = -2t + 2, z = -t + 1$.
(a) Show that ℓ_1 and ℓ_2 are skew lines.
(b) Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be parallel planes which contain ℓ_1 and ℓ_2 respectively. Find equations for $\mathcal{P}_1$ and $\mathcal{P}_2$.

answer: $\mathcal{P}_1 : y - 2z + 7 = 0$ and $\mathcal{P}_2 : y - 2z = 0$

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13. A line is the intersection of the two planes $y = -\frac{1}{3}x + \frac{8}{3}$ and $5y - z = 16$. Give a vector parametrization for the line.

answer: The cross product of normal vectors for the two planes $\langle 1, 3, 0\rangle \times \langle 0, 5, -1\rangle$ is a direction vector for the line, and in vector form the line is parametrized by $\vec{r}(t) = \langle -3t - 1, t + 3, 5t - 1\rangle$

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14. The planes $\mathcal{P}_1 : 3x - 5y + 2z = 1$ and $\mathcal{P}_2 : 3x - 5y + 2z = k$ are parallel (where k is an arbitrary real constant).
(a) A line intersects $\mathcal{P}_1$ perpendicularly at the point $(2, 1, 0)$, where does it intersect $\mathcal{P}_2$?
(b) Find all values of k for which the distance between the planes equals 3.

answer: (a) $((73 + 3k)/38, (43 - 5k)/38, (k - 1)/19)$ (b) $k = 1 \pm 3\sqrt{38}$

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15. Consider the curve $C : x = t^2 - t, y = 3t^3, z = 2t - 3$.
(a) Does C pass through the origin?
(b) Show that C does pass through the point $P = (0, 3, -1)$.
(c) Find a vector that is tangent to C at P.
(d) Give an equation for the plane which intersects C perpendicularly at P.

answer:
(a) No, the equations $0 = t^2 - t, 0 = 3t^3, 0 = 2t - 3$ have no solution for t.
(b) Take $t = 1$.
(c) $\mathbf{r}'(1) = \langle 1, 9, 2\rangle$
(d) $x + 9y + 2z = 25$

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16. Find parametric equations for the line tangent to $C : x = te^t, y = e^t, z = te^{t^2}$ at the point $(0, 1, 0)$.

answer: $x + 2y + 2e^2 z = 2e^4$

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17. An object moves in 3-space according to the vector function $\mathbf{r}(t) = \langle t^2, 5t, t^2 - 16t\rangle$.
(a) What is the velocity of the object at time t?
(b) What is the speed of the object at time t?
(c) When is the speed a minimum?

answer:
(a) $\mathbf{v}(t) = \mathbf{r}'(t) = \langle 2t, 5, 2t - 16\rangle$
(b) $s(t) = (8t^2 - 64t + 281)^{1/2}$
(c) When $t = 4$.

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18. A curve C is described parametrically by the vector function $\mathbf{r}(t) = \langle 2t, t^2, t\rangle$. Let $P = (-4, 4, -2)$ be the point on C where $t = -2$.
(a) Give a parametrization for the line ℓ tangent to C at P.
(b) Determine the speed and unit tangent vector at time t.

answer:
(a) $x = 2t - 4, y = -4t + 4, z = t - 2$
(b) $s(2) = |\mathbf{r}'(2)| = \sqrt{21}$ and the unit tangent vector at time $t = 2$ equals

$$\mathbf{r}'(2)/|\mathbf{r}'(2)| = \langle 2, -4, 1\rangle/\sqrt{21} = \langle 2/\sqrt{21}, -4/\sqrt{21}, 1/\sqrt{21}\rangle$$

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19. Is there a point on the curve C with scalar parametrization $x = t^2 - t, y = 3t^3, z = 2t - 3$ at which the tangent line to C is parallel to the line $x = t - 1, y = -2t + 1, z = 4t$? Explain.

answer: No. There is no value of t for which $\langle 2t - 1, 9t^2, 2\rangle$ is parallel to $\langle 1, -2, 4\rangle$