Brief Answers

**Math 2433** **Name:**
**Sample Exam 1**
**Fall 2021**

PROBLEM 1. *(20 points)* Let **a** be the vector represented by the arrow starting at $P = (2, 3, -5)$ and ending at $Q = (0, 4, -7)$. Let **b** be a vector with length 4 which forms an angle of $\pi/4$ with **a**.
(a) Express **a** in terms of the coordinate vectors **i**, **j** and **k**.
(b) Find the two unit vectors which are parallel to **a**.
(c) Determine $\mathbf{a} \cdot \mathbf{b}$.

(a) $\vec{a} = -2\vec{i} + \vec{j} - 2\vec{k}$

(b) $\pm \frac{1}{3}\langle -2, 1, -2 \rangle$

(c) $\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos(\pi/4)$
$= 3 \cdot 4 \cdot 1/\sqrt{2} = 12/\sqrt{2} = 6\sqrt{2}$

PROBLEM 2. *(20 points)* Let $\mathbf{a} = \langle 1, 0, 2 \rangle$ and $\mathbf{b} = \langle -2, 1, 3 \rangle$.
(a) Determine the magnitudes of **a** and **b** and the cosine of the angle $\theta$ between the two vectors.
(b) Find two unit vectors that are orthogonal to both **a** and **b**.
(c) What is the area of the parallelogram determined by **a** and **b**?

(a) $|\vec{a}| = \sqrt{5}$, $|\vec{b}| = \sqrt{14}$, $\cos\theta = \dfrac{\vec{a} \cdot \vec{b}}{\sqrt{5}\sqrt{14}} = \dfrac{4}{\sqrt{70}}$

(b) $\pm \frac{1}{\sqrt{54}}\langle 2, 7, -1 \rangle$

(c) $|\vec{a} \times \vec{b}| = \sqrt{54}$

$P \quad Q \quad R \quad S$

PROBLEM 3. *(10 points)* Determine whether or not the four points $(1, 1, 1)$, $(3, -1, 0)$, $(-1, 0, 2)$, $(7, 5, -2)$ and are coplanar in $\mathbb{R}^3$.

$\vec{a} = \overrightarrow{PQ}$

$\vec{b} = \overrightarrow{PR}$

$\vec{c} = \overrightarrow{PS}$

$\vec{a} \cdot (\vec{b} \times \vec{c}) = \langle 2, -2, -1 \rangle \cdot \langle -1, 0, -2 \rangle = 0$

So the 4 points are coplanar

PROBLEM 4. *(20 points)* Let $\mathbf{u} = \langle -6, 1, 3 \rangle$ and $\mathbf{v} = \langle 4, 0, -2 \rangle$.

(a) If $\mathbf{u} = \overrightarrow{PQ}$ and $Q = (10, -2, 7)$ then what is $P$?

(b) Determine the cosine of the angle $\theta$ between $\mathbf{u}$ and $\mathbf{v}$.

(c) Find the two unit vectors that are parallel to $\mathbf{v}$.

(d) Determine the vector projection $\mathbf{proj_u}(\mathbf{v})$ of $\mathbf{v}$ onto $\mathbf{u}$.

(e) (bonus) If the vector projection of $\mathbf{proj_b}(\mathbf{a})$ equals $\mathbf{b}$ what does that say about the relationship between $\mathbf{a}$ and $\mathbf{b}$.

(a) $P = (16, -3, 4)$

(b) $\cos\theta = -15/\sqrt{230}$

(d) $\frac{1}{23}\langle 90, -15, -45 \rangle$

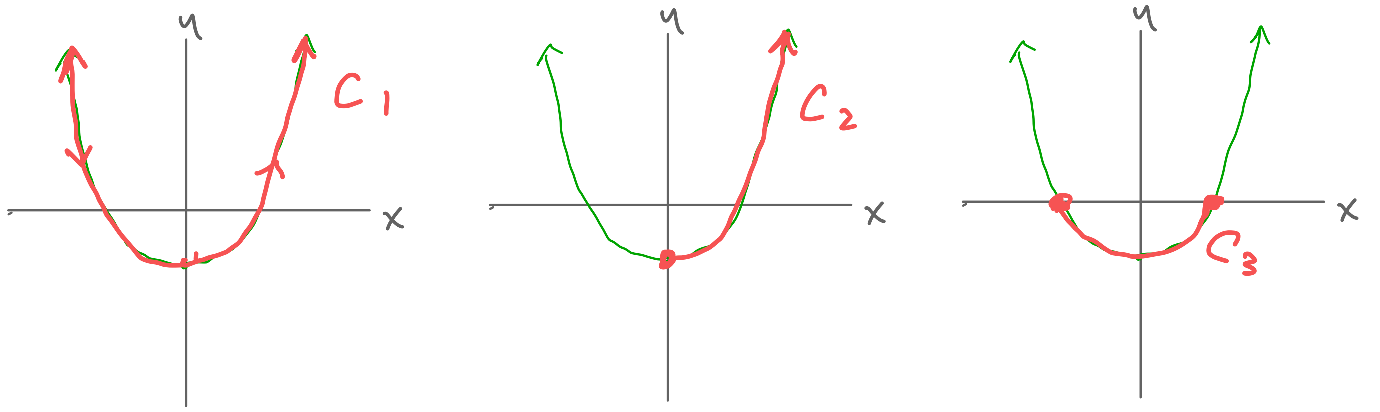
(e) one answer would be:

$\vec{a} \cdot (\vec{a} - \vec{b}) = 0$

Problem 5. *(15 points)* Three curves are described by parametrizations

$$C_1 : x = t, y = t^2 - 1, \qquad C_2 : x = t^2, y = t^4 - 1, \qquad C_3 : x = \cos(t), y = \cos^2(t) - 1 .$$

Draw separate pictures of the three curves and describe how they are related yet different.




All three curves lie on the parabola $y = x^2 - 1$.

Problem 6. *(20 points)* Consider the curve described by the parametric equations $x = t - t^2$, $y = t - t^3$.
(a) Does the curve pass through the point $(-2, 3)$? Explain.
(b) Find all points on the curve where the tangent line to the curve has slope 5. (Giving $t$-values is sufficient.)
(c) Determine $d^2y/dx^2$.

(a) No

(b) $\dfrac{dy}{dx} = \dfrac{1-3t^2}{1-2t}$. There are two points with $t = \frac{1}{3}(5 \pm \sqrt{13})$

(c) $\dfrac{d^2y}{dx^2} = \dfrac{6t^2 - 6t + 2}{(1-2t)^3}$

(The procedure for (c) is described in the class notes for 9-15.)

PROBLEM 7. *(20 points)* An object in motion in the plane is located at $(x, y) = (2t^3 + 3t^2 - 12t + 7, t^2 - 1)$ at time $t$ (where $-\infty < t < \infty$). Let $C$ be the curve that it traces out.

(a) Determine any points where $C$ crosses the $x$-axis.

(b) Find an equation for the line which is tangent to $C$ at the point where $t = 2$.

(c) For which values of $t$ is the object moving upward?

(d) For which values of $t$ is the object moving to the right?

(e) Use your answers to (c) and (d) to draw a rough picture of $C$.

(f) The curve $C$ has one point where it crosses itself. Find the $t$-values for that point.

(a) $(0,0)$ and $(20,0)$

(b) $y = \frac{1}{6}x + \frac{7}{6}$

(c) $t \geq 0$

(d) $t \leq -2$ and $t \geq 1$

(f) $t = -\sqrt{6}$ and $t = \sqrt{6}$. The point is $(25,5)$