Calculus II–Infinite Series Review Sheet

Convergence Tests:

<u>TEST FOR DIVERGENCE</u>. *If* $\lim_{n\to\infty} a_n \neq 0$ *then the series* $\sum_{n=1}^{\infty} a_n$ *diverges.*

<u>INTEGRAL TEST</u>. *Let* $f(x)$ *be a positive, continuous decreasing function for* $x \geq 1$ *and let* $a_n = f(n)$.
(a) If the improper integral $\int_1^{\infty} f(x)\,dx$ *converges then the series* $\sum_{n=1}^{\infty} a_n$ *converges.*
(b) If the improper integral $\int_1^{\infty} f(x)\,dx$ *diverges then the series* $\sum_{n=1}^{\infty} a_n$ *diverges.*

<u>COMPARISON TEST</u>. *Let* $\{a_n\}$ *and* $\{b_n\}$ *be sequences of positive numbers with* $a_n \leq b_n$ *for all positive integers* n.
(a) If $\sum_{n=1}^{\infty} b_n$ *converges then* $\sum_{n=1}^{\infty} a_n$ *converges.*
(b) If $\sum_{n=1}^{\infty} a_n$ *diverges then* $\sum_{n=1}^{\infty} b_n$ *diverges.*

<u>LIMIT COMPARISON TEST</u>. *Let* $\{a_n\}$ *and* $\{b_n\}$ *be sequences of positive numbers with* $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = c$ *where* $0 < c < \infty$.
(a) If $\sum_{n=1}^{\infty} b_n$ *converges then* $\sum_{n=1}^{\infty} a_n$ *converges.*
(b) If $\sum_{n=1}^{\infty} b_n$ *diverges then* $\sum_{n=1}^{\infty} a_n$ *diverges.*

<u>RATIO TEST</u>. *Suppose that* $a_n > 0$ *for all* n *and that* $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$.
(a) If $L < 1$ *then* $\sum_{n=1}^{\infty} a_n$ *converges.*
(b) If $L > 1$ *then* $\sum_{n=1}^{\infty} a_n$ *diverges. In fact, if* $L > 1$ *then* $\lim_{n\to\infty} a_n = \infty$.

<u>ROOT TEST</u>. *Suppose that* $a_n > 0$ *for all* n *and that* $\lim_{n\to\infty} \sqrt[n]{a_n} = L$.
(a) If $L < 1$ *then* $\sum_{n=1}^{\infty} a_n$ *converges.*
(b) If $L > 1$ *then* $\sum_{n=1}^{\infty} a_n$ *diverges .*

<u>ALTERNATING SERIES TEST</u>. *Let* $\{b_n\}$ *be a decreasing sequence of positive numbers with* $\lim_{n\to\infty} b_n = 0$. *Then the series* $\sum_{n=1}^{\infty} (-1)^n b_n$ *converges.*

<u>ABSOLUTE CONVERGENCE TEST</u>. *If* $\sum_{n=1}^{\infty} |a_n|$ *converges (that is, if* $\sum_{n=1}^{\infty} a_n$ *"converges absolutely") then* $\sum_{n=1}^{\infty} a_n$ *converges.*

A series $\sum_{n=1}^{\infty} a_n$ is said to **converge absolutely** if the positive series $\sum_{n=1}^{\infty} |a_n|$ converges. Note that the Integral, Comparison, Limit Comparison, Root and Ratio Tests are all tests that apply only to positive series (or really to any series that has only finitely many negative terms). However they can be used to determine the absolute convergence of any series.

IMPORTANT BASIC PRINCIPLE: An infinite series $\sum_{n=M}^{\infty} a_n$ will converge if and only if the series $\sum_{n=L}^{\infty} a_n$ converges. This means that the value of the starting index for the series has no effect on whether it converges or diverges. So it is common to leave off the indexing entirely and just say that $\sum a_n$ converges or diverges. (The same comments apply in like manner for absolute convergence and conditional convergence.) However, if you want to determine the sum of a convergent series then that does depend on where the indexing starts. For example, $\sum_{n=0}^{\infty}(2/3)^n = 3$ but $\sum_{n=1}^{\infty}(2/3)^n = 2$.