EXAM 1 **Name:**
Math 2924
9/4/19

*Instructions: To ensure getting full credit you must show your reasoning process on each problem.*

PROBLEM 1. *(25 points)* Let $g(x) = x^2 e^x$.
(a) Find all of the local extremes of $g$ and classify them as local max or local min.
(b) What is the range of the function $g$? Explain.

(a) Find $g'(x)$: $g'(x) = 2xe^x + x^2e^x = e^x(x+2)x$

Find critical numbers: $e^x(x+2)x = 0$

$(x+2)x = 0$

$x = -2$ or $x = 0$

The first derivative test:

$g' > 0$ $g' < 0$ $g' > 0$

$g$ inc $-2$ $g$ dec $0$ $g$ inc

$g(-2) = 4e^{-2}$ is a local maximum value of $g(x)$.
$g(0) = 0$ is a local minimum value of $g(x)$.

(b). Range $(g) = [0, \infty)$.
Firstly we know that $g(x) = x^2 e^x > 0$ [$(0, \infty)$ is in the domain.] because $x^2 \geq 0$ and $e^x > 0$. Secondly $0$ is included in the domain because $g(0) = 0$. Therefore the range of the function $g$ is $[0, \infty)$.

PROBLEM 2. *(20 points)* Let $F(x) = \ln(2x+3)$.

(a) Determine the domain of $F$.

(b) Determine the intervals of increase/decrease for $F$ and use the result to explain why this function has an inverse function.

(c) Find a formula for $F^{-1}(x)$.

(a) Every expression inside a log function needs to be positive.

$2x+3>0$, $x > -\frac{3}{2}$. Domain$(F) = (-\frac{3}{2}, \infty)$.

(b). $F'(x) = \dfrac{2}{2x+3} > 0$ for all $x > -\frac{3}{2}$.

The interval of increase for F is $(-\frac{3}{2}, \infty)$ and there is no interval of decrease for F.

F(x) is increasing over its whole domain. $\Rightarrow$ F(x) is a one-to-one function. $\Rightarrow$ F(x) has an inverse function.

(c) $y = \ln(2x+3) \Rightarrow e^y = 2x+3 \Rightarrow x = \dfrac{e^y - 3}{2}$

$F^{-1}(x) = \dfrac{e^x - 3}{2}$, $x \in (-\infty, \infty)$.

PROBLEM 3. *(15 points)* Calculate the value of $\int_0^{1/2} \frac{1}{1+4x^2}\,dx$

$u = 2x$

$du = 2dx$, $dx = \frac{1}{2}du$

$u(0) = 0$, $u(1/2) = 1$

$$\int_0^{1/2} \frac{1}{1+4x^2}\,dx = \int_0^1 \frac{1}{1+u^2}\,\frac{1}{2}du$$

$$= \frac{1}{2}\arctan(u)\Big|_{u=0}^{1}$$

$$= \frac{1}{2}\arctan(1)$$

$$= \frac{\pi}{8}$$

PROBLEM 4. *(15 points)* Find the derivative with respect to $x$ and simplify:
(a) $\arctan(3x)$, (b) $exp(x)^x$, (c) $x^{exp(x)}$

(a) $$\frac{d}{dx}[\arctan(3x)] = \frac{3}{1+(3x)^2} = \frac{3}{1+9x^2}$$

(b) $$\frac{d}{dx}[\exp(x)^x] = \frac{d}{dx}[e^{x^2}] = e^{x^2}\cdot 2x = 2xe^{x^2}$$

(c) $$\frac{d}{dx}[x^{\exp(x)}] = \frac{d}{dx}\left[e^{\ln(x^{\exp(x)})}\right] = \frac{d}{dx}[e^{e^x\cdot\ln x}]$$

$$= e^{e^x\cdot\ln x}\left(e^x\ln x + \frac{e^x}{x}\right)$$

$$= x^{\exp(x)}\left(e^x\ln x + \frac{e^x}{x}\right)$$

PROBLEM 5. *(15 points)* Determine the intervals of concavity for the function $f(x) = \ln(x^2+1)$ and indicate whether the function has any points of inflection.

Domain $(f) = (-\infty, \infty)$ because $x^2+1>0$ for all $x \in \mathbb{R}$.

First derivative $\quad f'(x) = \dfrac{2x}{x^2+1}$

Second derivative $\quad f''(x) = \dfrac{2(x^2+1) - 2x\cdot 2x}{(x^2+1)^2} = \dfrac{-2x^2+2}{(x^2+1)^2} = \dfrac{-2(x-1)(x+1)}{(x^2+1)^2}$

Critical numbers of $f'$ $\quad \dfrac{-2(x-1)(x+1)}{(x^2+1)^2} = 0$, $x=1$ or $x=-1$

| $f''<0$ | | $f''>0$ | | $f''<0$ |
|---|---|---|---|---|
| | $-1$ | | $1$ | |
| $f\cap$ | | $f\cup$ | | $f\cap$ |

$f(x)$ is concave up on $(-1,1)$ and is concave down on $(-\infty,-1)\cup(1,\infty)$.

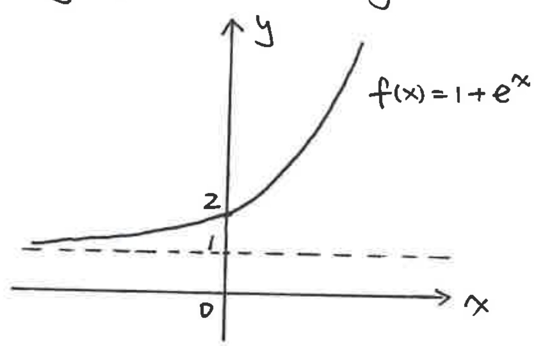
$f(x)$ has two points of inflections, $(-1, \ln 2)$ and $(1, \ln 2)$.

PROBLEM 6. *(15 points)* (a) Sketch the graph of $f(x) = 1 + e^x$ being sure to clearly identify any intercepts or asymptotes. (Its OK to base your answer on the known graph of $y = e^x$.)

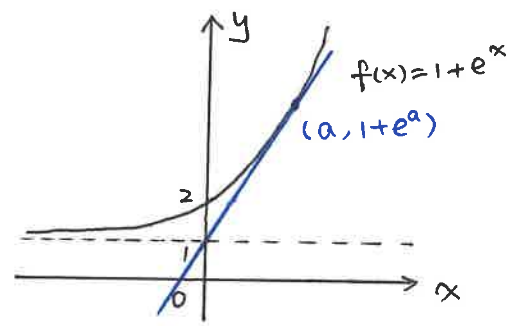
(b) By referring to the sketch in (a), how many points are there on the graph of $y = f(x)$ where the tangent line at that point has $y$-intercept equal to 1? (Might be good to redraw the picture.)

(c) Find the coordinates of any points on the graph of $y = f(x)$ for which the tangent line has $y$-intercept 1.

(a) graph of $y = e^x$ $\xrightarrow[\text{by 1 unit}]{\text{move upward}}$ graph of $f(x) = 1 + e^x$




(a)




(b)

(b) There is only one point of $y=f(x)$ where the tangent line at that point has $y$-intercept equal to 1. ( Blue Line )

Warm tip: Use a ruler to be a tangent line of $y=f(x)$. As you mov
the ruler along the curve to the right, you will notice:

- When $x<0$, the $y$-intercept of the tangent line will go from 1 to 2, i.e., $(1,2)$.
- When $x=0$, the $y$-intercept will be equal to 2.
- When $x>0$, the $y$-intercept will go from 2 to $-\infty$, which will go pass 1.

(c) Suppose the point $(a, 1+e^a)$ on the graph of $y=f(x)$ has tangent line with $y$-intercept equal to 1.

The slope of the tangent line is

$$f'(a) = e^x\big|_{x=a} = e^a.$$

The tangent line is

$$y - f(a) = f'(a)(x-a)$$

$$y - (1+e^a) = e^a(x-a)$$

$$y = e^a x - ae^a + e^a + 1$$

Because the tangent line has $y$-intercept 1, we have

$-ae^a + e^a + 1 = 1$, $(a-1)e^a = 0$, $a = 1$.

Therefore, $(1, 1+e)$ is the point on the graph of $y=f(x)$ for which the tangent line ha y-interc equal t 1.