EXAM 2
Math 2924
9/24/19

Name:

Instructions: To ensure getting full credit you must show your reasoning process on each problem.

PROBLEM 1. *(20 points)* (a) Calculate the integral $\int \sqrt{49 - x^2}\, dx$
(b) Use differentiation to verify that your answer to (a) is correct.

(a) $\int \sqrt{49 - x^2}\, dx$

$$\left[\begin{array}{l} x = 7\sin\theta \\ dx = 7\cos\theta\, d\theta \end{array}\right]$$

$$= \int \sqrt{49 - 49\sin^2\theta} \cdot 7\cos\theta\, d\theta$$

$$= \int 7\cos\theta \cdot 7\cos\theta\, d\theta$$

$$= 49 \int \cos^2\theta\, d\theta$$

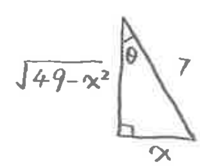
$$= 49 \int \frac{1}{2}(1 + \cos(2\theta))\, d\theta$$

$$= \frac{49}{2}\left(\theta + \frac{1}{2}\sin(2\theta)\right) + C$$

$$= \frac{49}{2}(\theta + \sin\theta\cos\theta) + C$$

$$= \frac{49}{2}\theta + \frac{49}{2}\sin\theta\cos\theta + C$$

$$\left[\sin\theta = \frac{x}{7}\right]$$



$$= \frac{49}{2}\arcsin\left(\frac{x}{7}\right) + \frac{49}{2} \cdot \frac{x}{7} \cdot \frac{\sqrt{49-x^2}}{7} + C$$

$$= \frac{49}{2}\arcsin\left(\frac{x}{7}\right) + \frac{x\sqrt{49-x^2}}{2} + C$$

(b) $\frac{d}{dx}\left[\frac{49}{2}\arcsin\left(\frac{x}{7}\right) + \frac{x\sqrt{49-x^2}}{2} + C\right]$

$$= \frac{49}{2} \frac{\frac{1}{7}}{\sqrt{1 - \left(\frac{x}{7}\right)^2}} + \frac{1}{2}\left(\sqrt{49-x^2} + x\frac{-2x}{2\sqrt{49-x^2}}\right)$$

$$= \frac{7}{2} \frac{1}{\frac{\sqrt{49-x^2}}{7}} + \frac{1}{2}\sqrt{49-x^2} - \frac{x^2}{2\sqrt{49-x^2}}$$

$$= \frac{49}{2\sqrt{49-x^2}} + \frac{1}{2}\sqrt{49-x^2} - \frac{x^2}{2\sqrt{49-x^2}} = \frac{49-x^2}{2\sqrt{49-x^2}} + \frac{1}{2}\sqrt{49-x^2} = \sqrt{49-x^2} \checkmark$$

PROBLEM 2. *(15 points)* Find the length of the segment of the parabola $y = x^2$ between $(0,0)$ and $(1/2, 1/4)$.

$$\int_0^{\frac{1}{2}} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_0^{\frac{1}{2}} \sqrt{1 + (2x)^2}\, dx$$

$$= \int_0^{\frac{1}{2}} \sqrt{1 + 4x^2}\, dx$$

$$= \int_0^{\frac{\pi}{4}} \sqrt{\sec^2\theta} \cdot \frac{1}{2}\sec^2\theta\, d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}} \sec^3\theta\, d\theta$$

$$\left[\begin{array}{l} 2x = \tan\theta \\ x = \frac{1}{2}\tan\theta \\ dx = \frac{1}{2}\sec^2\theta\, d\theta \\ \theta(0) = 0 \\ \theta(\frac{1}{2}) = \frac{\pi}{4} \end{array}\right]$$

$$= \frac{1}{4}\left(\sec\theta\tan\theta + \ln\left|\sec\theta + \tan\theta\right|\right)\Big|_{\theta=0}^{\frac{\pi}{4}} = \frac{1}{4}\left(\sqrt{2} + \ln(\sqrt{2}+1)\right)$$

PROBLEM 3. *(15 points)* Work the indefinite integrals:

(a) $\displaystyle\int \frac{x^2+2x+2}{x+2}\,dx = \int \frac{x(x+2)+2}{x+2}\,dx = \int x + \frac{2}{x+2}\,dx = \frac{x^2}{2} + 2\ln|x+2| + C$

(b) $\displaystyle\int \frac{x+2}{x^2+2x+2}\,dx = \int \frac{(x+1)+1}{x^2+2x+2}\,dx = \underbrace{\frac{1}{2}\int \frac{2(x+1)}{x^2+2x+2}\,dx}_{\mathrm{I}} + \underbrace{\int \frac{1}{(x+1)^2+1}\,dx}_{\mathrm{II}}$

I $\quad \dfrac{u = x^2+2x+2}{du = (2x+2)dx} \quad \displaystyle\frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|x^2+2x+2| + C = \frac{1}{2}\ln(x^2+2x+2) + C$

II $= \arctan(x+1) + C$ $\quad$ Therefore, $\displaystyle\int \frac{x+2}{x^2+2x+2}\,dx = \frac{1}{2}\ln(x^2+2x+2) + \arctan(x+1) + C$

(c) $\displaystyle\int \frac{e^{2x}}{1+e^x}\,dx$

$$\frac{u = 1+e^x}{du = e^x dx}$$

$= (u-1)\,dx$

$dx = \frac{1}{u-1}\,du$

$$\int \frac{(u-1)^2}{u}\,\frac{1}{u-1}\,du = \int \frac{u-1}{u}\,du = \int 1 - \frac{1}{u}\,du$$

$$= u - \ln|u| + C = 1 + e^x - \ln|1+e^x| + C$$

$$= e^x - \ln(1+e^x) + C$$

PROBLEM 4. *(15 points)*

(a) Calculate the indefinite integral $\int \sec^6(x)\,dx$.

(b) Use integration by parts to find a formula describing $\int \sec^7(x)\,dx$ in terms of $\int \sec^5(x)\,dx$.

(a) $$\int \sec^6(x)\,dx = \int \sec^4(x)\cdot\sec^2(x)\,dx$$

$$= \int (1+\tan^2(x))^2\cdot\sec^2(x)\,dx \qquad \left[\begin{array}{l} u=\tan x \\ du=\sec^2(x)\,dx \end{array}\right]$$

$$= \int (1+u^2)^2\,du$$

$$= \int 1+2u^2+u^4\,du$$

$$= u+\frac{2u^3}{3}+\frac{u^5}{5}+C$$

$$= \tan x+\frac{2\tan^3(x)}{3}+\frac{\tan^5(x)}{5}+C$$

(b) $$\int \sec^7(x)\,dx$$

$$= \int \sec^5(x)\cdot\sec^2(x)\,dx$$

$$\left[\begin{array}{l} u=\sec^5(x) \\ dv=\sec^2(x)\,dx \\ du=5\sec^4(x)\sec x\tan x\,dx \\ \quad =5\sec^5(x)\tan x\,dx \\ V=\tan x \end{array}\right]$$

$$= \sec^5(x)\tan x-\int \tan x\,5\sec^5(x)\tan x\,dx$$

$$= \sec^5(x)\tan x-5\int \sec^5(x)\tan^2(x)\,dx$$

$$= \sec^5(x)\tan x-5\int \sec^5(x)(\sec^2(x)-1)\,dx$$

$$= \sec^5(x)\tan x-5\int \sec^7(x)\,dx+5\int \sec^5(x)\,dx$$

$$\int \sec^7(x)\,dx = \frac{1}{6}\left[\sec^5(x)\tan x+5\int \sec^5(x)\,dx\right]$$

PROBLEM 5. *(20 points)* Consider the rational function

$$R(x) = \frac{3x^3 + x + 1}{x^2(1+x^2)}.$$

(a) Give the general form of the partial fraction decomposition of $R(x)$.

(b) Use your answer to (a) to find $\int R(x)\,dx$.

(c) Give the general form of the partial fraction decomposition of $(R(x))^2$.

(a) $R(x) = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{1+x^2}$, A, B, C and D are real numbers.

(b) $R(x) = \frac{Ax(1+x^2) + B(1+x^2) + (Cx+D)x^2}{x^2(1+x^2)}$

$$\begin{array}{ll} x^3: & \\ x^2: & \\ x: & \\ 1: & \end{array} \left\{ \begin{array}{l} A + C = 3 \\ B + D = 0 \\ A = 1 \\ B = 1 \end{array} \right. \Rightarrow \left\{ \begin{array}{l} A = 1 \\ B = 1 \\ C = 2 \\ D = -1 \end{array} \right.$$

$$\int R(x)\,dx = \int \frac{1}{x} + \frac{1}{x^2} + \frac{2x-1}{1+x^2}\,dx$$

$$= \ln|x| - \frac{1}{x} + \ln(1+x^2) - \arctan(x) + C$$

(c) $(R(x))^2 = \frac{(3x^3+x+1)^2}{x^4(1+x^2)^2}$

$$= \frac{E}{x} + \frac{F}{x^2} + \frac{G}{x^3} + \frac{H}{x^4} + \frac{Ix+J}{1+x^2} + \frac{Kx+L}{(1+x^2)^2}$$

where E, F, G, H, I, J, K and L are real numbers.

PROBLEM 6. *(15 points)* Determine the limits:

(a) $\lim_{x\to 0}(1-2x)^{3/x}$

Set $y = (1-2x)^{3/x}$

$\ln y = \frac{3}{x}\ln(1-2x)$

$= \frac{3\ln(1-2x)}{x}$

$\lim_{x\to 0}\ln y = \lim_{x\to 0}\frac{3\ln(1-2x)}{x}$

$\overset{L'H}{=} \lim_{x\to 0}\frac{3\,\frac{-2}{1-2x}}{1}$

$= -6$

Therefore, $\lim_{x\to 0}(1-2x)^{3/x} = e^{-6}$.

(b) $\lim_{x\to 1+}\frac{x^7-1}{x^5-1}$

$\overset{L'H}{=} \lim_{x\to 1^+}\frac{7x^7}{5x^5}$

$= \frac{7}{5}$

(c) $\lim_{x\to 1+}(\ln(x^7-1)-\ln(x^5-1))$

$= \lim_{x\to 1^+}\ln\left(\frac{x^7-1}{x^5-1}\right)$

$= \ln\left(\lim_{x\to 1^+}\frac{x^7-1}{x^5-1}\right)$

$\overset{(b)}{=} \ln\left(\frac{7}{5}\right)$

PROBLEM 7. *(5 points)*

State the addition identities for sine and cosine:

$\sin(A+B) =$ $\sin A\cos B + \cos A\sin B$

$\cos(A+B) =$ $\cos A\cos B - \sin A\sin B$