EXAM 3 **Name:**
Math 2924
10/23/19

Instructions: To ensure getting full credit you must show your reasoning process on each problem.

PROBLEM 1. *(20 points)* Explain why each of the integrals is improper and then compute their value

(a) $\displaystyle\int_0^4 \frac{dx}{4-x}$ (b) $\displaystyle\int_0^2 \frac{dx}{\sqrt{4-x^2}}$

(a) The integral $\int_0^4 \frac{dx}{4-x}$ is improper because $\frac{1}{4-x}$ is not defined at $x = 4$.

$$\int_0^4 \frac{dx}{4-x} = \lim_{t\to 4^-} \int_0^t \frac{dx}{4-x} = \lim_{t\to 4^-} \left[-\ln|4-x|\right]\Big|_0^t$$

$$= \lim_{t\to 4^-} \left[-\ln(4-t) + \ln(4)\right] = \infty$$

(b) The integral $\int_0^2 \frac{dx}{\sqrt{4-x^2}}$ is improper because $\frac{1}{\sqrt{4-x^2}}$ is not defined at $x = 2$.

$$\int_0^2 \frac{dx}{\sqrt{4-x^2}} = \lim_{t\to 2^-} \int_0^t \frac{dx}{\sqrt{4-x^2}} = \lim_{t\to 2^-} \int_0^t \frac{dx}{2\sqrt{1-\left(\frac{x}{2}\right)^2}}$$

$$= \lim_{t\to 2^-} \int_0^t \frac{\frac{1}{2}}{\sqrt{1-\left(\frac{x}{2}\right)^2}}\,dx = \lim_{t\to 2^-} \left[\arcsin\left(\frac{x}{2}\right)\Big|_0^t\right]$$

$$= \lim_{t\to 2^-} \arcsin\left(\frac{t}{2}\right) = \arcsin(1) = \frac{\pi}{2}.$$

PROBLEM 2. *(20 points)* Consider the cardiod $r = 1 - 3\cos(\theta)$.

(a) Find the slope of the tangent line at the point where $\theta = \pi/4$.

(b) Find the θ values $0 \le \theta \le 2\pi$ for all points where the tangent is horizontal.

(a) Rectangular coordinates,
$$\begin{cases} x(\theta) = r\cos\theta = (1-3\cos\theta)\cos\theta \\ y(\theta) = r\sin\theta = (1-3\cos\theta)\sin\theta \end{cases}$$

$$\left.\frac{dy}{dx}\right|_{\theta=\pi/4} = \left.\frac{dy/d\theta}{dx/d\theta}\right|_{\theta=\pi/4} = \left.\frac{3\sin^2\theta + (1-3\cos\theta)\cos\theta}{3\sin\theta\cos\theta + (1-3\cos\theta)(-\sin\theta)}\right|_{\theta=\pi/4}$$

$$= \frac{3\cdot\left(\frac{\sqrt{2}}{2}\right)^2 + \left(1-\frac{3\sqrt{2}}{2}\right)\frac{\sqrt{2}}{2}}{3\cdot\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2} + \left(1-\frac{3\sqrt{2}}{2}\right)\left(-\frac{\sqrt{2}}{2}\right)} = \frac{\frac{\sqrt{2}}{2}}{3-\frac{\sqrt{2}}{2}} = \frac{\sqrt{2}}{6-\sqrt{2}} = \frac{3\sqrt{2}+1}{17}$$

The slope of the tangent line at the point where $\theta = \pi/4$ is $\frac{3\sqrt{2}+1}{17}$.

(b) The cardiod $r = 1-3\cos\theta$ has horizontal tangent line when $dy/d\theta = 0$,

i.e., $3\sin^2\theta + (1-3\cos\theta)\cos\theta = 0$

With the help of Mathematica, $\theta \approx 0.65, 2.25, 4.03$ and 5.63.

PROBLEM 3. *(20 points)* Let C be the curve traced by an object moving in the plane according the parametric equations $x = t^3 - 3t, y = t^2$.

(a) Determine the t-intervals on which the object is moving right-left and up-down.

(b) Find any points on the curve where C crosses itself.

(c) Plot the most important points (including those from parts (a) and (b)) and sketch the curve. Be sure to clearly identify any intercepts.

(d) Determine the speed of the object at time t and express the distance that the object has traveled from time $t = 0$ to $t = 3$ as an integral.

(e) At what time or times is the object moving the slowest?

(a) $\frac{dx}{dt} = 3t^2 - 3 = 3(t+1)(t-1)$

moving right $\Rightarrow \frac{dx}{dt} > 0 \Rightarrow 3(t+1)(t-1) > 0 \Rightarrow t > 1$ or $t < -1$.

moving left $\Rightarrow \frac{dx}{dt} < 0 \Rightarrow$ Complement of above $\Rightarrow -1 < t < 1$

$\frac{dy}{dt} = 2t$

moving up $\Rightarrow \frac{dy}{dt} > 0 \Rightarrow t > 0$

moving down $\Rightarrow t < 0$.

(b) $\begin{cases} t^3 - 3t = s^3 - 3s & ① \\ t^2 = s^2 & ② \end{cases}$

② $\Rightarrow t = \pm s \xrightarrow{t \neq s} t = -s \xrightarrow[①]{\text{plug into}} -s^3 + 3s = s^3 - 3s$

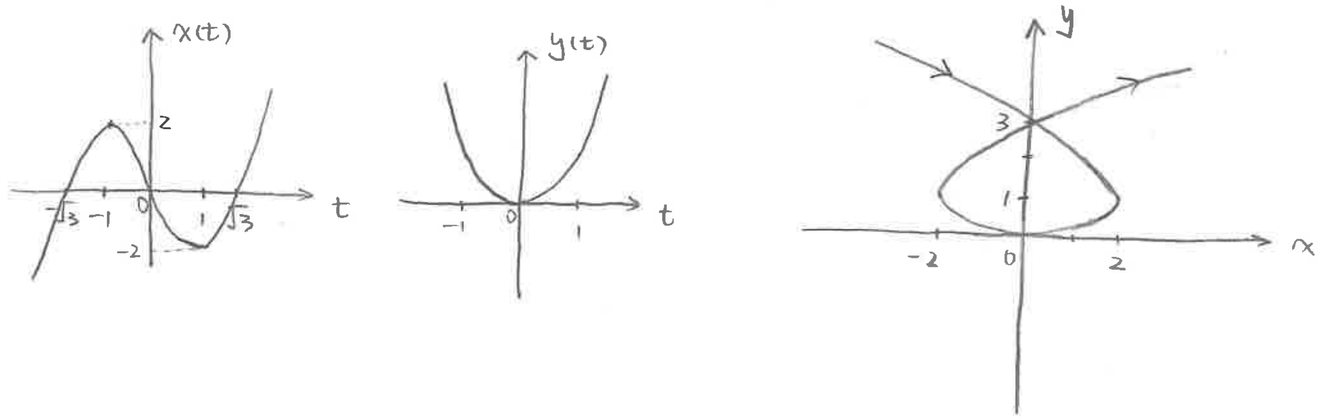
$2s^3 - 6s = 0$, $s^3 - 3s = 0$, $s(s^2 - 3) = 0$, $s_1 = 0$ ($\times$ b/c $s = t = 0$)

$s_2 = \sqrt{3}$ and $s_3 = -\sqrt{3}$.

Therefore the curve crosses itself when $t = \sqrt{3}$ and $t = -\sqrt{3}$

at $(x(\sqrt{3}), y(\sqrt{3})) = (x(-\sqrt{3}), y(-\sqrt{3})) = (0, 3)$.

(c)



(d) speed $= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(3t^2-3)^2 + (2t)^2}$

$= \sqrt{9t^4 - 18t^2 + 9 + 4t^2}$

$= \sqrt{9t^4 - 14t^2 + 9}$

distance that the object has traveled from $t=0$ to $t=3$:

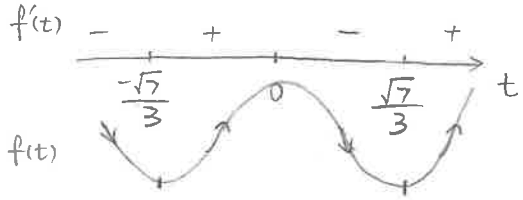
$\int_0^3 \sqrt{9t^4 - 14t^2 + 9}\, dt$

(e) Denote speed by $f(t) = \sqrt{9t^4 - 14t^2 + 9}$.

$f'(t) = \dfrac{36t^3 - 28t}{2\sqrt{9t^4 - 14t^2 + 9}}$

critical numbers: $f'(t) = 0 \Rightarrow 36t^3 - 28t = 0 \Rightarrow t(9t^2 - 7) = 0$

$t = 0$ or $t = \pm\frac{\sqrt{7}}{3}$.



Since $f(t)$ is an even function, i.e. $f(t) = f(-t)$, the object moves the slowest when $t = \pm\frac{\sqrt{7}}{3}$.

PROBLEM 4. *(20 points)* Consider the polar curve $C: r = \sin(3\theta)$.

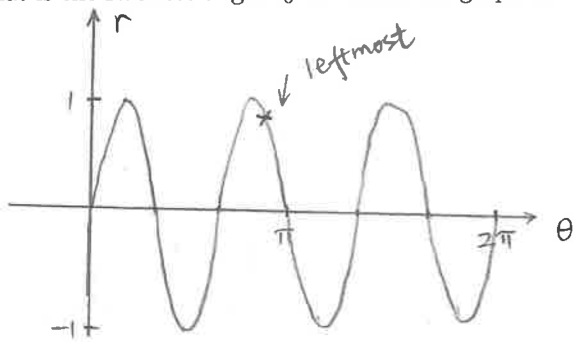
(a) Sketch graph of $r = \sin(3\theta)$ in the (θ, r)-plane.

(b) Sketch C.

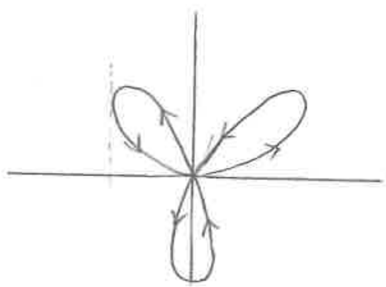
(c) Determine the polar coordinates of the leftmost point on C.

(d) What is the smallest angle θ_0 for which the graph of $r = \sin(3\theta)$ for $0 \le \theta \le \theta_0$ is the entire curve C?

(a)



(b)



(c) $\frac{dx}{d\theta} = 0$ gives vertical tangent lines.

$x = r\cos\theta = \sin(3\theta)\cos\theta$

$\frac{dx}{d\theta} = 3\cos(3\theta)\cos\theta - \sin(3\theta)\sin\theta$

$3\cos(3\theta)\cos\theta - \sin(3\theta)\sin\theta = 0$

With the help of Mathematica, $\theta = \pi n - \tan^{-1}(\sqrt{6-\sqrt{33}})$, $n \in \mathbb{Z}$.

As we can see from the graphs the leftmost point occurs when θ is between $\frac{2\pi}{3}$ and π. By choosing $n=1$, $\theta \approx 2.67$.

The leftmost point on C is $(\sin(3\theta_1), \theta_1)$, where $\theta_1 = \pi - \tan^{-1}(\sqrt{6-\sqrt{33}})$, or approximately, $(\sin(8.01), 2.67) \approx (0.99, 2.67)$

(d) $\theta_0 = \pi$. The curve C has period π.

Recall that $(r, \theta) = (-r, \theta + \pi)$. Suppose (R, α) is on C. Because $\sin(3(\alpha+\pi)) = \sin(3\alpha + 3\pi) = -\sin(3\alpha) = -R$, the point (R, α) moves back to itself after rotating counterclockwise by π.

PROBLEM 5. *(15 points)* Calculate the sums:

(a) $\Sigma_{k=0}^{\infty}\frac{(-2)^k}{5^k}$ (b) $\Sigma_{k=0}^{\infty}\frac{(-2)^{k+3}}{5^{k-2}}$ (c) $\Sigma_{k=10}^{\infty}\frac{(-2)^{k+3}}{5^{k-2}}$ (d) $\Sigma_{k=0}^{10}\frac{2^k}{5^k}$ $= S_{10}$

(a)
$$= \frac{1}{1-\left(\frac{-2}{5}\right)} = \frac{5}{7}$$

(b)
$$= \frac{(-2)^3}{5^{-2}}\sum_{k=0}^{\infty}\left(\frac{-2}{5}\right)^k = -8\cdot 25\cdot\frac{5}{7} = \frac{-1000}{7}$$

(c)
$$= \frac{(-2)^3}{5^{-2}}\sum_{k=10}^{\infty}\left(\frac{-2}{5}\right)^k = -200\cdot\frac{\left(\frac{-2}{5}\right)^{10}}{1-\left(\frac{-2}{5}\right)} = -2^3\cdot 5^2\,\frac{\frac{2^{10}}{5^{10}}\cdot 5}{7} = -\frac{2^{13}}{7\cdot 5^7}$$

(d)
$$S_{10}\cdot\left(1-\frac{2}{5}\right) = \left(\sum_{k=0}^{10}\frac{2^k}{5^k}\right)\left(1-\frac{2}{5}\right)$$
$$= \left(1+\frac{2}{5}+\cdots+\frac{2^{10}}{5^{10}}\right)-\left(\frac{2}{5}+\cdots+\frac{2^{11}}{5^{11}}\right)$$
$$= 1-\frac{2^{11}}{5^{11}}$$
$$S_{10} = \frac{1-\frac{2^{11}}{5^{11}}}{1-\frac{2}{5}}.$$
$$= \frac{5-\frac{2^{11}}{5^{10}}}{3}$$

PROBLEM 6. *(5 points)* State the addition formula for $\cos(A+B)$.

$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$

PROBLEM 7. *(5 points)* A curve C is described by parametric equations $C: x = x(t), y = y(t)$. What properties does C satisfy in each of the follwing cases:

(a) Both $x(t)$ and $y(t)$ are odd functions.

(b) Both $x(t)$ and $y(t)$ are even functions.

(a) $x(t)$ and $y(t)$ are odd functions. $\Rightarrow \begin{cases} x(-t) = -x(t) \\ y(-t) = -y(t) \end{cases}$

$\Rightarrow$ The curve C is symmetric about the origin. Moreover, it passes through the origin.

(b) $x(t)$ and $y(t)$ are even functions. $\Rightarrow \begin{cases} x(-t) = x(t) \\ y(-t) = y(t). \end{cases}$

$\Rightarrow$ The curve C stops at $(x(0), y(0))$ and then retraces its steps.