EXAM 3 **Name:**
Math 2924
11/25/19

PROBLEM 1. *(25 points)* Use convergence tests to establish each of the following. Be sure to explain why all necessary hypotheses are met.

(a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges conditionally.

- $\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges, because this is a p-series with $p=1$.
- Consider $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$. $\lim_{n\to\infty} \frac{1}{n} = 0$ & $\frac{1}{n+1} < \frac{1}{n}$ for all $n \geq 1$

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges by alternating series test.

In conclusion, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges conditionally.

(b) $\sum_{n=0}^{\infty} \frac{\sqrt{n^3+4n+2}}{3n^3+1}$ converges.

Consider the convergent p-series $\sum_{n=0}^{\infty} \frac{1}{n^{3/2}}$ $(p = 3/2)$.

$$\lim_{n\to\infty} \left| \frac{\sqrt{n^3+4n+2}}{3n^3+1} \cdot n^{3/2} \right| = \lim_{n\to\infty} \left| \frac{\sqrt{n^6+4n^4+2n^3}}{3n^3+1} \right| = \frac{1}{3}$$

$\sum_{n=0}^{\infty} \frac{\sqrt{n^3+4n+2}}{3n^3+1}$ converges by limit comparison test.

(c) $\sum_{n=1}^{\infty} \frac{n^n}{n}$ diverges.

$$\lim_{n\to\infty} \frac{n^n}{n} = \lim_{n\to\infty} n^{n-1} = \infty$$

$\sum_{n=1}^{\infty} \frac{n^n}{n}$ diverges by test for divergence.

(d) $\sum_{n=1}^{\infty} \frac{n}{n^n}$ converges.

$\lim_{n\to\infty} \sqrt[n]{\frac{n}{n^n}} = \lim_{n\to\infty} \frac{n^{\frac{1}{n}}}{n} = 0 < 1$, $\sum_{n=1}^{\infty} \frac{n}{n^n}$ converges by root test.

Or

$\lim_{n\to\infty} \frac{n+1}{(n+1)^{n+1}} \cdot \frac{n^n}{n} = \lim_{n\to\infty} \frac{n+1}{n} \cdot \left(\frac{n}{n+1}\right)^n \cdot \frac{1}{n+1} = 1 \cdot e^{-1} \cdot 0 = 0 < 1$,

$\sum_{n=1}^{\infty} \frac{n}{n^n}$ converges by ratio test.

PROBLEM 2. *(25 points)* Consider the series $\sum_{n=1}^{\infty}(-1)^n \frac{n}{n^2+1}\frac{x^n}{9^n}$.

(a) Determine the radius of convergence of the power series.

(b) Determine all values of x for which the series converges.

(c) Is the function $f(x) = \sum_{n=1}^{\infty}(-1)^n \frac{n}{n^2+1}\frac{x^n}{9^n}$ increasing or decreasing at $x = -1$? Explain.

(a) $$\left| (-1)^{n+1}\frac{n+1}{(n+1)^2+1}\frac{x^{n+1}}{9^{n+1}} \cdot \frac{1}{(-1)^n} \cdot \frac{n^2+1}{n} \frac{9^n}{x^n} \right| = \frac{n+1}{n} \cdot \frac{n^2+1}{(n+1)^2+1} \cdot \frac{1}{9}|x| \xrightarrow{n\to\infty} \frac{1}{9}|x|$$

$\frac{1}{9}|x| < 1 \Rightarrow |x| < 9$. The radius of convergence of the power series is 9

(b). Consider $x=-9$. The series becomes $\sum_{n=1}^{\infty}(-1)^n \frac{n}{n^2+1}\frac{(-9)^n}{9^n} = \sum_{n=1}^{\infty}\frac{n}{n^2+1}$

We apply limit comparison test with $\sum_{n=1}^{\infty}\frac{1}{n}$.

$\lim_{n\to\infty}\frac{n}{n^2+1}\cdot n = \lim_{n\to\infty}\frac{n^2}{n^2+1} = 1$. $\sum_{n=1}^{\infty}(-1)^n\frac{n}{n^2+1}\frac{x^n}{9^n}$ diverges if $x=-9$.

If $x=9$. The series is $\sum_{n=1}^{\infty}(-1)^n\frac{n}{n^2+1}\frac{9^n}{9^n} = \sum_{n=1}^{\infty}(-1)^n\frac{n}{n^2+1}$.

We apply alternating series test.

- $\lim_{n\to\infty}\frac{n}{n^2+1} = 0$ ✓
- $\left(\frac{n}{n^2+1}\right)_{n=1}^{\infty}$ is a decreasing sequence. ✓

$\sum_{n=1}^{\infty}(-1)^n\frac{n}{n^2+1}\frac{x^n}{9^n}$ is convergent when $x=9$.

The interval of convergence is $(-9, 9]$.

(c) $f'(x) = \sum_{n=1}^{\infty}(-1)^n\frac{n}{n^2+1}\frac{nx^{n-1}}{9^n} = \sum_{n=1}^{\infty}(-1)^n\frac{n^2}{n^2+1}\frac{x^{n-1}}{9^n}$

$f'(-1) = \sum_{n=1}^{\infty}(-1)^n\frac{n^2}{n^2+1}\frac{(-1)^{n-1}}{9^n} = \sum_{n=1}^{\infty}(-1)^{2n-1}\frac{n^2}{9^n(n^2+1)} = -\sum_{n=1}^{\infty}\frac{n^2}{9^n(n^2+1)} < 0$

$f(x)$ is decreasing at $x=-1$.

PROBLEM 3. *(10 points)* Describe the domain of the function $g(x) = \sum_{n=1}^{\infty}(-1)^n\frac{n}{n^2+1}\frac{(x+3)^n}{9^n}$. (HINT: Compare with previous problem.)

Note that $g(x) = f(x+3)$.

$f(x)$ has domain $(-9, 9]$, so $f(x+3)$ has domain

$$-9 < x+3 \le 9$$
$$-12 < x \le 6$$

i.e. $g(x)$ has domain $(-12, 6]$.

PROBLEM 4. *(25 points)* Give Maclaurin series expansions for each of the following:

(a) $\dfrac{1}{1-x} \quad = \quad \sum_{n=0}^{\infty} x^n$

(b) $\dfrac{1}{1+x^3} \quad = \quad \dfrac{1}{1-(-x^3)} = \sum_{n=0}^{\infty} (-x^3)^n = \sum_{n=0}^{\infty} (-1)^n x^{3n}$

(c) $-\dfrac{3x^2}{(1+x^3)^2} = \dfrac{d}{dx}\left\{\dfrac{1}{1+x^3}\right\} = \dfrac{d}{dx}\left\{ \sum_{n=0}^{\infty} (-1)^n x^{3n} \right\} = \sum_{n=1}^{\infty} (-1)^n \dfrac{d}{dx}(x^{3n}) = \sum_{n=1}^{\infty} (-1)^n\, 3n\, x^{3n-1}$

(d) $e^{-2x^3} \quad = \quad \sum_{n=0}^{\infty} \dfrac{(-2x^3)^n}{n!} = \sum_{n=0}^{\infty} \dfrac{(-1)^n 2^n x^{3n}}{n!}$

(e) $\displaystyle\int e^{-2x^3}\,dx \quad = \quad \int \sum_{n=0}^{\infty} \dfrac{(-1)^n 2^n x^{3n}}{n!}\,dx = \sum_{n=0}^{\infty} \dfrac{(-1)^n 2^n}{n!} \int x^{3n}\,dx = \sum_{n=0}^{\infty} \dfrac{(-1)^n 2^n}{n!} \dfrac{x^{3n+1}}{3n+1} + C$

(f) $x^2 \sin(x) \quad = \quad x^2 \sum_{n=0}^{\infty} \dfrac{(-1)^n \cdot x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \dfrac{(-1)^n \cdot x^{2n+3}}{(2n+1)!}$

PROBLEM 5. *(10 points)* Given that $\ln(2) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ calculate the value of $\sum_{n=2}^{\infty}\left(2\frac{(-1)^{n+1}}{n} + 5\frac{(-1)^n}{n}\right)$.

$$\sum_{n=2}^{\infty}\left(2\frac{(-1)^{n+1}}{n} + 5\frac{(-1)^n}{n}\right)$$

$$= 2\boxed{\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{n}} + 5\sum_{n=2}^{\infty}\frac{(-1)^n}{n}$$

$$= 2\left(\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} - \frac{(-1)^2}{1}\right) - 5\boxed{\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{n}}$$

$$= 2\left(\ln(2) - 1\right) - 5\left(\ln(2) - 1\right)$$

$$= -3\ln(2) + 3$$

PROBLEM 6. *(10 points)* Consider the convergent infinite series $\sum_{n=1}^{\infty} \frac{2}{n^2+n}$.

(a) Write the first three terms s_1, s_2 and s_3 of the sequence of partial sums of the series.

(b) Find a general formula for the nth partial sum s_n and use it to determine the sum of the series. (HINT: Before writing out s_n, use partial fractions to rewrite $\frac{2}{n^2+n}$.)

(a) $S_1 = a_1 = \frac{2}{1+1} = 1$, $S_2 = a_1 + a_2 = 1 + \frac{2}{2^2+2} = \frac{4}{3}$,

$S_3 = S_2 + a_3 = \frac{4}{3} + \frac{2}{3^2+3} = \frac{3}{2}$

(b) $a_n = \frac{2}{n^2+n} = \frac{2}{n(n+1)} = \frac{2}{n} - \frac{2}{n+1}$

$$S_n = a_1 + a_2 + a_3 + a_4 + \cdots + a_n$$

$$= \left(\frac{2}{1} - \cancel{\frac{2}{2}}\right) + \left(\cancel{\frac{2}{2}} - \cancel{\frac{2}{3}}\right) + \left(\cancel{\frac{2}{3}} - \cancel{\frac{2}{4}}\right) + \left(\cancel{\frac{2}{4}} - \cancel{\frac{2}{5}}\right) + \cdots + \left(\cancel{\frac{2}{n}} - \frac{2}{n+1}\right)$$

$$= 2 - \frac{2}{n+1}$$

$$\sum_{n=1}^{\infty}\frac{2}{n^2+n} = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(2 - \frac{2}{n+1}\right) = 2.$$