Exam 2 – Some Review Problems
Math 2924

1. Let $f(x)$ be the rational function $f(x) = \dfrac{2x}{(x-1)^2(x+2)}$.
(a) Determine the partial fractions decomposition for $f(x)$.
(b) Use your answer to (a) to calculate $\int f(x)\,dx$.
(c) Check your answer in part (b).

ANSWER:
(a) The partial fractions decomposition is $f(x) = \dfrac{4/9}{x-1} + \dfrac{2/3}{(x-1)^2} + \dfrac{-4/9}{x+2}$.
(b) $\displaystyle\int f(x)\,dx = \frac{4}{9}\ln|x-1| - \frac{2}{3}\frac{1}{x-1} - \frac{4}{9}\ln|x+2| + C$
(c) Check by differentiating: if the answer in (b) is correct then its derivative will equal $\dfrac{2x}{(x-1)^2(x+2)}$.

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2. Determine the limits:
(a) $\displaystyle\lim_{x\to 0}\frac{\tan^2(x)}{x}$
(b) $\displaystyle\lim_{x\to 0+}\ln(x)\sin(x)$
(c) $\displaystyle\lim_{x\to(\pi/2)+}\tan(x)^{\cos(x)}$
(d) $\displaystyle\lim_{x\to\infty}(1+3/x)^{5x}$

3. If $x = 4\sec(\theta)$ express $\sin(\theta)$ and $\tan(\theta)$ in terms of x by using a right triangle analysis.

ANSWER:
$\sin(\theta) = \dfrac{\sqrt{x^2-16}}{x}$ and $\tan(\theta) = \dfrac{\sqrt{x^2-16}}{4}$

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4. State the half angle formulas for the cosine function and for the sine function.

5. Determine the limits: (a) $\displaystyle\lim_{x\to 0}\frac{\ln(x+1)}{\tan(3x)}$ (b) $\displaystyle\lim_{x\to 0}(x+1)^{1/\tan(3x)}$

6. Use integration by parts taking $u = \ln(x)$ to work out the integral $\int x^p\ln(x)\,dx$ where $p > 0$.

7. Show that the integral $\int_0^\infty xe^{1-x^2}\,dx$ converges and determine its value.

8. Consider the integral $\displaystyle\int\frac{x^3}{\sqrt{9-x^2}}\,dx$.
(a) Calculate the integral using a trig substitution.
(b) Calculate the integral using a u-substitution.
(c) Show that your answers in (a) and (b) agree.

9. (a) Determine the partial fraction decomposition of the rational function

$$R(x) = \frac{4x^2+x+4}{(x-1)(x^2+x+1)}$$

(b) Use (a) to determine $\int R(x)\,dx$.

10. For which real numbers k is the rational function $\dfrac{x+1}{(2x^2+kx+5)^3}$ a partial fraction? Explain.

11. The rational function $Q(x)$ given below can be written as the sum of a polynomial $P(x)$ and partial fractions. What does $P(x)$ equal?

$$Q(x) = \frac{5x^5 - 12x^4 + 5x^3 + 5x^2 - 7x + 2}{x^4 - 2x^3 + x - 1}$$

12. Would the form "∞^0" be considered to be determinate or indeterminate? Write a sentence or two justifying your answer.

13. Work the following integrals. (Be sure to clearly identify the method of approach.)

(a) $\displaystyle\int x\sec^2(x)\,dx$

(b) $\displaystyle\int \frac{2}{x^2+4x+5}\,dx$

(c) $\displaystyle\int \sin^2(t)\cos^2(t)\,dt$

(d) $\displaystyle\int \frac{1}{(\sqrt{49-x^2})^3}\,dx$

(e) $\displaystyle\int \frac{1}{(\sqrt{x^2-49})^3}\,dx$

ANSWER:

(a) $\displaystyle\int x\sec^2(x)\,dx = x\tan(x) + \ln|\cos(x)| + C$ using integration by parts ($u = x, dv = \sec^2(x)\,dx$).

(b) $\displaystyle\int \frac{2}{x^2+4x+5}\,dx = 2\tan^{-1}(x+2) + C$ by completing the square $x^2+4x+5 = (x+2)^2+1$ and substituting $u = x+2, du = dx$.

(c) $\displaystyle\int \sin^2(t)\cos^2(t)\,dt = \frac{t}{8} - \frac{\sin(4t)}{32} + C$ using the half-angle identities $\sin^2(\theta) = \frac{1}{2}(1-\cos(2\theta))$ and $\cos^2(\theta) = \frac{1}{2}(1+\cos(2\theta))$.

(d) $\displaystyle\int \frac{1}{(\sqrt{49-x^2})^3}\,dx = \frac{x}{49\sqrt{49-x^2}} + C$ using the trig substitution $x = 7\sin(\theta), dx = 7\cos(\theta)\,d\theta$.

(e) $\displaystyle\int \frac{1}{(\sqrt{x^2-49})^3}\,dx = -\frac{x}{49\sqrt{x^2-49}} + C$ using the trig substitution $x = 7\sec(\theta), dx = 7\sec(\theta)\tan(\theta)\,d\theta$.

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14. Let C be the curve segment which is the graph of $y = \frac{2}{3}(x^2+1)^{3/2}$ for $0 \le x \le 2$. Describe the arclength of C as a definite integral and compute its value.

ANSWER:

$\displaystyle\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{4x^4+4x^2+1} = 2x^2+1$ and $\displaystyle L(C) = \int_0^2 2x^2+1\,dx = 22/3.$

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15. Determine the integral $\int \cos(\ln(x^2))\,dx$.

ANSWER:

Using integration by parts twice gives $\displaystyle\int \cos(\ln(x^2))\,dx = \frac{1}{5}x\cos(\ln(x^2)) + \frac{2}{5}x\sin(\ln(x^2)) + C$

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16. The integral $\int \ln(2x)dx$ can be worked out using integration by parts taking $dv = dx$. Carry this out.

17. Give the general form of the partial fractions decomposition of the rational function $\displaystyle\frac{x^4-1}{x(2x+1)^3(x^2+x+1)^2}$.

18. Determine the precise partial fractions decomposition for $f(x) = \dfrac{x+1}{(x-1)(x^2+1)}$ and then determine $\int f(x)dx$.

19. Work the indefinite integrals:

(a) $\displaystyle\int \tan(e^x)e^x dx$

(b) $\displaystyle\int \frac{x^3}{x^2-2x+1}dx$

(c) $\displaystyle\int \sin^4(x)\cos^3(x)dx$

(d) $\dfrac{1+2x^3}{x^2\sqrt{x^2-1}}dx$

20. Let $\mathcal{R}$ be the region below $y = e^{-2x}$ and inside the first quadrant.

(a) Express the area of $\mathcal{R}$ as an improper integral and a limit of definite integrals, then compute it.

(b) Determine the volume of the solid obtained by rotating $\mathcal{R}$ around the x-axis.

21. Determine the indefinite integral $\displaystyle\int \ln(x^2+1)dx$.

22. Give the general form of the partial fractions decomposition for the following rational functions.

(a) $\dfrac{x^4-3x+1}{(x-3)^3(x^2-x+6)^2}$

(b) $\dfrac{x^4-3x+1}{(x-3)^3(x^2-x-6)^2}$

23. If $\sec(\theta) = x/5$ use a right triangle analysis to express the following as functions of x:

(a) $\sin(\theta)$ (b) $\cos(\theta)$ (c) $\sin(2\theta)$ (d) $\tan(2\theta)$

24. Determine the limit $\displaystyle\lim_{x\to 0} \frac{e^{3x}-1-3x-9x^2/2}{x^3}$. (Write and explain your steps carefully!)

25. Find the arclength of the curve $y = 2x^{3/2}$, with $0 \le x \le b$. What value should you choose for b so that the arclength is equal to 2?

26. Let $F(x) = x^{1/x}$ for $x > 0$.

(a) Does the graph of $y = F(x)$ have a horizontal asymptote as x goes to ∞? If so what is it?

(b) Does the graph of $y = F(x)$ have a right-side vertical asymptote at $x = 0$?

(c) Does $F(x)$ have any local extremes?